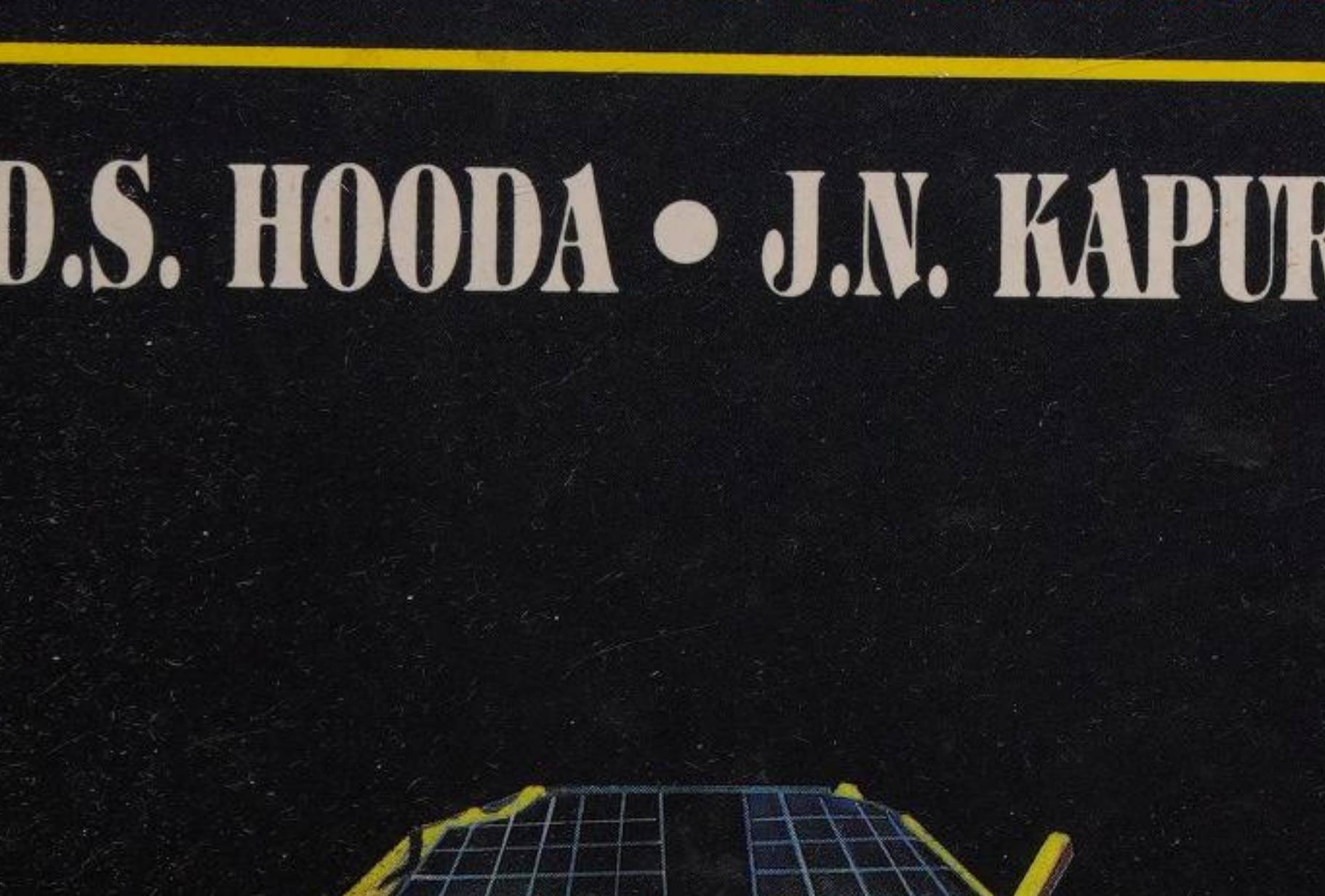
ĀRYABHAṬA

Life and Contributions

D.S. HOODA • J.N. KAPUR



NEW AGE INTERNATIONAL PUBLISHERS

SHOUT
SHARE
IT

ĀRYABHATA

Life and Contributions

D.S. HOODA • J.N. KAPUR

PUBLISHING FOR ONE WORLD

NEW AGE INTERNATIONAL (P) LIMITED, PUBLISHERS

Formerly WILEY EASTERN LIMITED

New Delhi • Bangalore • Bombay • Calcutta • Guwahati • Hyderabad

Lucknow • Madras • Pune • London • Bangkok

PREFACE

We are happy to present this book on the life and work of Āryabhata, who was one of the greatest mathematicians and astronomers of all times, and certainly the first of the great mathematicians and astronomers responsible for the renaissance of mathematics and science in ancient India.

It is true that Indian mathematics and astronomy had great achievements to their credit even before Āryabhata's time, but contributions in these fields were usually made by groups of saint-scholars, who combined in their writings mathematics, astronomy, astrology, religious rituals, science and spiritual knowledge, in a more or less integrated manner.

Āryabhata, for the first time, had the courage to break with this tradition. He studied thoroughly and completely all the mathematics and astronomy known in his time, systematized it, found the gaps in the knowledge of these disciplines and filled these gaps with his own researches. He had the courage to reject the accepted values of astronomical parameters, if these were inconsistent with his and others' observations. In this sense, he was a true scientist uninhibited by any consideration, other than that of search for truth.

He was dedicated to Lord Brahma, the creator of

this universe, and he pursued mathematics and astronomy as he believed that researches in these disciplines would enable him to understand fully the universe created by his Lord. Instead of just worshiping Nature, he wanted to understand it and in this approach, he was very near in spirit to Kepler, Newton and later mathematicians and astronomers.

Āryabhata ushered in a revolution in Indian mathematics, astronomy and science, the effects of which were felt and continued to be for eight centuries and even after that.

Had political condition been more stable, Indian scientists would probably have continued to follow the path shown by Āryabhata, and India would have continued as a leader in the world of science and technology during the later centuries as well.

In this book, we have examined Āryabhata's life and work. To put his work into proper perspective, in the first chapter, we give a brief history of ancient Indian mathematics and astronomy, pointing out their great strengths as well as some of their weaknesses.

In the second chapter, we have put down all that is known about his life, which, however is not very much. There are certain controversies about his date and place of birth, and we have discussed all of these, giving different points of view. We have also discussed about his education, profession and his devotion to Brahma.

In the third chapter, we discuss the contents of the *Āryabhatiya*, the great work of the master, written at the very early age of 23 years. The style and contents of this book continue to inspire the succeeding generations of mathematicians and astronomers in India.

In the fourth and fifth chapters, we discuss Āryabhata's most significant contributions to mathematics and astronomy. In the last chapter, we discuss the

relevance of Āryabhata to modern mathematics and science.

There are already hundreds of articles and a few books on Āryabhata's life and work. The books are either short of popular accounts for the lay man, or detailed scholarly commentaries on Āryabhata's work. This book fills in a gap, since it is neither so short as to be sketchy, nor so detailed as to be of interest only to scholars in the history of mathematics and astronomy.

We believe that this book will be useful to all students and teachers of mathematics and science in all schools, colleges and universities in India. We also believe that it will be equally useful to all educated persons, in India and abroad, who are interested in knowing about ancient India's glory in the fields of mathematics and science.

This book would be able to inspire young students, raise their scientific morale and give them faith in themselves and in their ancient heritage. This inspiration would come as much from Āryabhata's work, as from his great scientific attitude of creativity, objectivity, a relentless and single minded pursuit of truth, his courage in rejecting all false statements and from his desire to understand Nature. We hope the great scientific values cherished by Āryabhata would continue to inspire all of us.

We are grateful to the publisher for their suggestions and help, and their interest in the publication of this book.

D.S. HOODA
J.N. KAPUR

CONTENTS

Preface *iii*

1. History of Ancient Indian Astronomy and Mathematics **1**

1.1 Introduction 1
1.2 A Brief History of Ancient Astronomy 4
1.3 An Assessment of Ancient Indian Astronomy 10
1.4 A Brief History of Ancient Indian Mathematics 12
1.5 Development of Mathematics in Various Countries 18
1.6 Tributes to Ancient Indian Mathematics 20
1.7 Concluding Remarks 25

2. Life History of Āryabhata **28**

2.1 Introduction 28
2.2 Birth 33
2.3 Kusumpura of Āryabhata 38
2.4 Teacher and Devotee of Brahma 40
2.5 Pupils of Āryabhata 42

3. The *Āryabhatiya* **45**

3.1 Introduction 45
3.2 A Collection of Two Compositions 48

3.3 Notable Points of the *Āryabhatiya* 52
3.4 Commentaries on the *Āryabhatiya* 54
3.5 The *Āryabhata-Siddhanta* 58
3.6 The *Āryabhatiya* and *Āryabhata-Siddhanta* 59
3.7 The *Surya-Siddhanta-Prakash* 62

4. Contributions of Āryabhata in Mathematics 63
4.1 Introduction 63
4.2 Arithmetic 64
4.3 Algebra 78
4.4 Value of π 82
4.5 Mensuration and Geometry 84
4.6 Trigonometry 88
4.7 Construction of Circles and the Shadow Sphere 92

5. Contributions of Āryabhata in Astronomy 97
5.1 Introduction 97
5.2 Main Contributions 99
5.3 *Bhagola* 105
5.4 *Khagola* 108
5.5 Spherical Astronomy 109
5.6 Eclipses of the Moon and the Sun 112
5.7 Planets Determined from Observations 116

6. Relevance of Āryabhata to Modern Mathematics and Science 118
6.1 Renaissance of Scientific Methodology Inspired by Āryabhata 118
6.2 Some Lessons from Āryabhata's Life and Work 121
6.3 Concluding Remarks 126

References **128**

CHAPTER 1

HISTORY OF ANCIENT INDIAN ASTRONOMY AND MATHEMATICS

1.1 Introduction

The history of every civilization is always shrouded in mystery and is full of controversy. The conclusions of a historian can always be challenged seriously on the grounds of insufficient data and knowledge, while the conclusions of a scientist can be challenged only on the grounds of accuracy, interpretation and validity of his experimental work and theory.

Civilization is spread over distant lands and languages. Therefore, no one can be perfectly familiar with all the available literature in different languages. We can only see whether one has really mastered the meaning and the spirit of different texts, and has not been misled by the translations and interpretations, given by his predecessors, to write the history of any aspect of an ancient civilization. But this is not an easy task.

It is also not easy to be completely objective. Thus, while some western mathematicians and historians of mathematics have been full of great praise for the contributions of ancient Indian mathematicians? (see section 1.5), and a few others have not been able to give them due credit for all their achievements. Their

opinions have been criticised by other scholars, both Indian and foreigners, who have regarded these opinions as unfair and biased.

To put the record straight, we shall try in this chapter to give a balanced view of the great achievements (and some shortcomings) of ancient Indian mathematics and astronomy. In the last chapter we shall discuss the relevance of ancient Indian mathematics and in particular the relevance of Āryabhata's contributions to modern mathematics and science. This will help to put Āryabhata's achievements in mathematics and astronomy in the proper perspective with respect to the contributions of his predecessors and successors in India, and to show how his contributions marked a turning point and ushered in a new renaissance of mathematics and astronomy in India.

It is sometimes stated that in ancient India no science ever attained an independent existence. All sciences were developed with the primary objective of helping the *Vedic* rituals. It is also said, sometimes, that any further pursuit of science was somewhat discouraged by the *Vedic* Hindus, who feared that it might prove to be a hindrance to their quest for knowledge of the Supreme, by diverting the mind to other cultural channels. It, perhaps, may be true that in the (earlier part of the *Vedic* age), science grew as a support to religion. Religion had been the prime concern of the ancient Hindus, and (mathematics grew as help to it and was kept subsidiary). But in the later part of the *Vedic* Age, all the sciences outgrew with their original purpose and were cultivated for their own sake.

The importance of mathematics was thus realized during the later part of *Vedic* Age. The culture of mathematics was not considered a hindrance to spiritual knowledge. In fact, it was considered to be a helpful adjunct to spiritual knowledge. In fact, it was considered to be a helpful adjunct to spiritual knowl-

edge. The importance of the culture of *Ganita* (mathematics) is also mentioned by the *jainas* whose religious literature is generally classified into four branches known collectively as *Anuyoga* (exposition of principles). One of them is *ganitanuyoga* (the exposition of the principles of mathematics). The knowledge of arithmetic and astronomy is stated to be one of the principal accomplishments of the *Jaina* priests. In Buddhist literature, too, arithmetic is regarded as the first and the noblest of the arts. This gives us a fair account of the importance and value attached to the culture of *ganita* in ancient India.

This chapter is addressed to those who are not specialists in the history of mathematics and astronomy, but who are aware that ancient Indians made important contributions to astronomy and mathematics, and are anxious to know what these contributions were and who made these contributions. We shall, therefore, give very brief history of ancient Indian astronomy and mathematics, and make some objective assessments of the contributions made.

It has already been pointed out that life in ancient India was dominated by religion. Mathematics was created for designing *yagnakunds* and altars for various types of *havans*. Astronomy was developed to regulate religious rituals. More mathematics was created to meet the needs of this astronomy. Most of Hindu astronomy was given in religious books and most of Hindu mathematics was given in books on Hindu astronomy.

The close relation between mathematics and astronomy has continued in modern times, though mathematics has many users other than astronomers (and religion does not need mathematics very much now). Moreover, modern astronomy also depends on many other disciplines like physics, chemistry, biology,

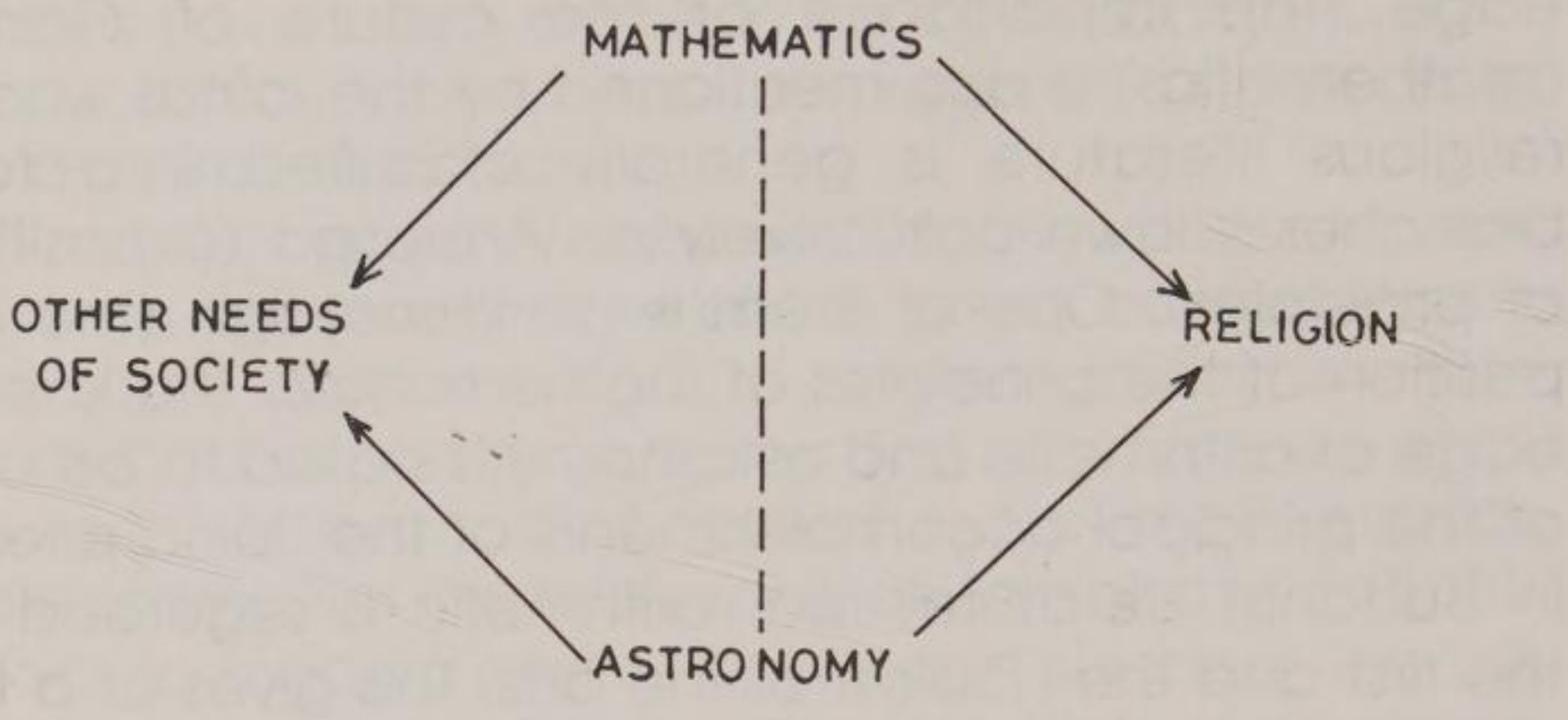


Fig. 1

etc. (which had not developed to the same extent as mathematics in ancient India).

1.2 Brief History of Ancient Indian Astronomy

For convenience, this history is divided into two periods: (a) The Pre-*Siddhanta* period (b) The *Siddhanta Jyotish* period.

The Pre-*Siddhanta* Period is further divided into four parts.

(i) Pre-Vedic astronomy

Though there is no direct reference to astronomy in the antiquities found in the excavations of Mohenjodaro and Harappa, yet it is believed that a civilization so highly developed could not do without a calender based on astronomy.

(ii) Vedic period astronomy

There are many astronomical references in the *vedas*, particularly the *Yajurveda*, which show that a system of astronomy, based on practical observations of

moon and stars, existed for providing the calendar to enable the priests to perform *yagnas,* which had to be performed in different parts of the year and at different points of the day. A day was reckoned from sunrise to sunset and the night was reckoned from sunset to sunrise. Each was divided into fifteen *mahurats*. These *mahurats* were named differently for each day and night of the dark and bright fortnights of a lunar month. Besides these, there were three divisions of a day *(Purvahna, Madhyahna* and *Sayahna)* and five divisions of a day (*Pratah, Sangava, Madhyama, Aparahna* and *Sayahna*) for rough division of time. The month used was the lunar month and all thirty days of the lunar month were given specific names. There were two groups of *Vedic* people, one reckoned a month from full moon to full moon, and the other from new moon to new moon. A year considered of twelve months (*Madhu, Madhava, Shukra, Shuchi, Nabha, Nabhasya, Isha, Urja, Sahas, Sahasya, Tapas* and *Tapasaya*), spread over six seasons (*Vasanta, Grishma, Varsha, Sharad, Hemanta* and *Shishira*). Later, with the discovery of *Nakshatras,* the names of months were changed to their present names, *Chaitra, Vaisakh,* etc. The solar year was also started to be used along with the lunar month, and a thirteenth month *Amhaspati* was introduced to regulate calculations.

There were two *Ayans* in a year *Uttarayana* and *Dakshinayana,* marking the period of northward and southward motion of the sun. A *Vishwa* day (an equinoxal day) is mentioned between the two *ayanas.* A *yuga* is also mentioned, but it is meant only for a period of five years in the *Vedic* period. Within the belt along which the Moon moves around the path of the Sun (ecliptic), wherein its maximum celestial longitude is five degrees north or south, a *Nakshatra* system was supposed to consist of 27 *Nakshatras,* all of which

were named in *Vedic* times, only slightly different from their present names. Among the planets, *Brahspati* (Jupiter), *Shukra* (Venus) and *Rahu* are mentioned, but it is not easy to see how those who constantly watched the *Nakshatras*, could have left the other planets unnoticed. Comets and meteors (*Dhumketu* and *Ulkas*) are clearly mentioned. There is no mention of a seven-day week or zodiac signs (*rashis*) in the *Vedas*.

(iii) Vedanga Jyotish period of astronomy

For every *veda*, rules were given for calculating astronomical results in a text called *Vedanga Jyotish*, supposed to be written by the unknown writer Leghadha. The period might be about the thirteenth century B.C. A period of five years was called *yuga* and the names of the five years were *Samvastar, Parivastar, Idavastar, Anuvastar* and *Idvastar*. A solar year was supposed to consist of 366 days. A *yuga* was supposed to consist of 1830 civil days, sixty two (synodic) months of moon, thirty omitted *tithis*, two intercalary months and five mean revolutions of the sun. *Vedanga Jyotish* dealt only with rough calculations of solar and lunar elements. There were no calculations for planets.

(iv) Mahabharata period of astronomy

During this period, Indians had knowledge of 5 planets, they observed them in the framework of *Nakshatras* belt and in conjunction with one another. They knew about the retrograde motions of planets and they attached astrological significance to all these phenomena. They had a time scale of *Kalpayuga* of 4.32×10^9 years and of a *Mahayuga* of 4.32×10^6 years, consisting of *Kalyuga* of 4,32,000 years, *Dwapu-*

ryuga of 8,64,000 years, *Treta yuga* of 12,96,000 years and *Satuga* of 17,28,000 years.

We now come to the *Siddhanta Jyotish* period of Indian astronomy. *Siddhanta Jyotish* means a system which gives rules for calculating time, motions of planets, etc., from the beginning of the universe (*Sreshtiadi*) to the end (*pralya*). A book is called *Karana* if it gives rules for calculating astronomical events from a fixed epoch, other than the beginning of *yuga* of *Kalpa*. The period in which these books were written is called the *Siddhanta Jyotish* period.

There are following five *Siddhanta* books which are supposed to be written either by immortal beings or by sages.

1. *Paithamaha* or *Brahma Siddhanta*
2. *Vasishtha Siddhanta*
3. *Romaka Siddhanta*
4. *Paulisa Siddhanta*
5. *Surya Siddhanta*

The most respected and scientific of these is the *Surya Siddhanta*, and its calculations are based in recorded coordinates of *Nakshatras*. It is believed to have been written in about 280 AD.

It is claimed that the third and fourth of the *Siddhanta* books may have foreign influence since these have, at some places, greater resemblance to Greek astronomy than to Indian astronomy. However, the most accurate of these, viz. the *Surya-Siddhanta* is certainly Indian. All these five *Siddhantas* were edited by the famous astronomer *Varaha Mihira*, in his famous work *Pancha Siddhantika*.

These texts dealt with the motion of heavenly bodies, and gave kinematic rules for the determination of the position of the Sun, the Moon and the Planets, at any desired time. The motion was explained in terms of epicycles and deferents. Each body was supposed to move uniformly along a circle. (A circle called

epicycle whose centre itself moved uniformly along another circle B, called deferent). Fairly accurate calculations could be made by suitable adjustments of A and B, and the position of the centre of B relative to the earth. In further refinement the centre of B itself was supposed to move uniformly along another circle C. These *Siddhantas* studied solar and lunar eclipses, conjunctions of planets, conjunctions of planets with *nakshatras*, the rising and setting of planets, phases of the moon, astronomical instruments, the creation of the universe, etc.

There are other books on astronomy, written later, by outstanding mathematicians and astronomers.

1. *Āryabhatiya*: It was written by Āryabhata 1(421 *Shaka*), whose fifteen hundredth anniversary we celebrated in 1976, and after whom we named our first satellite. This was the first book on astronomy supposed to be written by a non-divine being. The book also contains chapters on arithmetic, algebra and trigonometry, and gives the first tables for trigonometric functions. Details about this book are given in Chapter III '*The Āryabhatiya*'.

2. *Brahma Sphuta Siddhanta* (revised system of Brahma) by Brahmagupta (550 *Shaka*): This book gives accurate rules for finding positions of planets, and corrects errors in earlier works. Bhaskaracharya considered Brahmagupta as *Ganit Chakra Churamani* first gem among the circle of mathematicians.

3. *Baswati Karan* by Shatanand (1021 *Shaka*): This book uses centimal multiplication and division system.

4. *Siddhanta Shiromani* by Bhaskaracharya (1036 *Shaka*): This book is the most comprehensive and gives a complete account of Indian astronomy till that time. It also discusses the obliquity of the ecliptic and its effect on the equation of time. This contains four

parts, namely *Lilavati, Bijaganita, Goladhaya* and *Grahaganita*.

5. *Grahalagyam* by Ganesh (1442 *Shaka*): This book is useful for those who have only limited knowledge of mathematics.

6. *Siddhanta-Tattave Viveka* by Kakalaka Bhatt (1530 *Shaka*): It is a good book on mathematical astronomy.

We now mention some books written in Kerala.

1. *Deva Keralam, Sukra Keralam,* of about the seventh century AD.

2. *Sarkaranarayaneeyam,* by Sarkaranarayana of the ninth century AD. This book refers to an astronomical observatory and its instruments established by a contemporary king Ravi Varma.

3. *Venwaroham* by Mahadevan, discusses trigonometry and planetary motions.

4. *Karan Paddhati,* written in 1431 AD.

5. *Sadratnamala* by Sakara Vaiman, written in about 1530 AD.

6. *Tantra Samgraham* by Nilkantha, written in about 1500 AD.

7. *Yukti Bhasa* by Brahmadatta, written in 1639, AD.

The last five books are in verse form and give infinite series expressions for sin x, cos x and tan x about two and half centuries earlier than in Europe. The series known as Gregory's series for $\tan^{-1} x$ was also known in India more than two centuries earlier than in Europe.

Since astronomy is a practical subject, its success depends on the use of instruments. Ancient Indian astronomers were aware of this aspect and designed suitable instruments for astronomical measurements. In almost every book on astronomy, a chapter was devoted to the construction and use of such instruments.

The earliest instrument available in Vedic times was *Turya,* consisting of a quadrant circle with a movable tube through the centre and resting on the arc of the

quadrant, which is graduated in degrees and minutes. Through the tube, the heavenly bodies were observed, and the angle made by the tube with the vertical gave the zenith distance called *Natansk* and its complement *unantash* or altitude. During the *Vedanga Jyotish* period, a water clock was also developed.

Chapter XIII of *Surya Siddhanta* discusses the construction of an auxiliary sphere, a *Sankuyantra* or gnomon, the *Cakrayantra* or the graduated wheel, *Kapal Yantra* or hollow hemisphere, *Dhanus Yantra* or semicircle, *Naryantra, Mayur Yantra, Vanaryantra,* etc. Brahmagupta discusses, in *Brahma Sphuta Siddhanta,* seventeen instruments by which observations could be made. Bhaskaracharya gave the description of an additional instrument called *Phalak Yantra*. Another instrument called *Dhruvmrey Yantra* was devised by Padmanabhan in the fourteenth century. *Yantra Cintamani,* written by Chakardan, was a separate book on instruments.

An observatory appears to have been constructed at Ujjain or Avanti; it may have been at the same place where the present observatory stands. Another observatory, *Man Mandir,* was constructed by Raja Man Singh at Varanasi, and a number of instruments were added to it by Raja Jai Singh who also built the observatory at Jaipur as well as the city of Jaipur around it. He described all instruments except *rasivlaya,* for which an effort was made by Garret in 1907 to reconstruct it. However, in the process, he changed it completely. Raja Jai Singh also built the observatory at Delhi.

1.3 An Assessment of Ancient Indian Astronomy

We start by giving a quotation from 0. Neugebauer's

'*A History of Ancient Mathematical Astronomy*', a scholarly work comprising 3 volumes and more than 1200 pages, published in 1976.

> In spite of the pioneering work done by H.T. Colebrooke (1765-1837), G. Thibault (1848-1914) and others, the study of Hindu astronomy is still at its beginning. The mass of uninvestigated manuscript material in India as well as in western countries is enormous. May it suffice to remark that many hundreds of planetary tables are easily accessible in American libraries? So far only a preliminary study of this material has been made revealing a great number of parameters for lunar planetary tables. The planetary tables are of great extent and based on methods so far not encountered in western materials, the basic idea being that the planetary positions are computed for a whole year as function of the initial conditions at the entry of the Sun into Aries. When these methods were developed, we do not know, the extent texts suggest the date of the fourteenth century AD.

This statement clearly establishes the need for a massive research effort. It also is a testimonial to the originality of Indian work on astronomy.

There is a feeling that there has been some Greek influence on Indian astronomy, and two of the *Siddhantas* show this influence. However, the *Surya Siddhanta*, which is purely Indian, is far superior to them, and this was due entirely to the great mastery of ancient Indians over arithmetic and algebra. The Greeks were superior in geometry, but astronomy required computations, and therefore the Greeks were not proficient in astronomy.

The independence of the two developments is testified by the fact that the parameters used are

different; the methods of correction are different, the styles of presentation are different and mathematical backgrounds are somewhat different. Moreover, Indian astronomy was influenced much more by the needs of astrology. This is also testified by the fact that in Indian astronomy, Ujjain, the important centre of Indian astronomical activities, was taken as having zero longitude.

Astronomy is a practical subject, and truth in this field is obtained by successive approximations. The Hindus were always ready to give up the methods which gave wrong results, and were prepared to improve them. Starting with *Panch Siddhantika*, each succeeding work gives more accurate methods than the earlier ones.

Indian astronomy ultimately exercised greater influence on the development of modern Western astronomy, than on Greek or Babylonian systems, because of the influence of Arab astronomical works on Western scholars. Of special importance in Arabic astronomy was the *Khandakhadyaka* of Brahmagupta, written in 665 AD, and known to the Arabs as the *Al-Arkand*. Parameters from this work and Indian planetary theories found their way, through Arabic scholars, in particular through al-Khwarizmi, into Western European astronomy. The use of the zero meridian of 'Arin' (Ujjain) is the most obvious example of this influence.

1.4 A Brief History of Ancient Indian Mathematics

(i) *Sulva Sutras*: There are seven *Sulva Sutras* known by the name of *Bodhayana, Apasthamba, Katyayanam, Manava, Maitrayana, Varaha* and *Vidhuta*. They belong to the period 800 BC to 500 BC. They explain a large number of simple geometrical constructions imparting knowledge of the following

theorems: (a) the diagonals of a rectangle divide it into two equal parts, (b) the diagonals of a rectangle bisect each other, (c) the perpendicular, drawn from the vertex of an isosceles triangle to its base, bisects it, (d) a rectangle and a parallelogram on the same base and between the same parallels are equal in area, (e) the diagonals of a rhombus bisect each other at right angles, (f) properties of similar rectilinear figures (g) the theorem known after the name of Pythagoras i.e. Pythagoras theorem which states that the square of the hypotenuse is equal to the sum of squares of base and perpendicular. Irrational numbers were first used here and the approximation to $\sqrt{2}$ as $1 + 1/3 + 1/3.4 + 1/3.4.34$ was given. Problems in the construction of altars which required the solution of simultaneous linear equations in integers were solved and illustrated.

(ii) Jain mathematics belongs to the period from about 500 BC. to 100 BC. It gives elementary formulae for mensuration, approximate values of square roots (e.g. $\sqrt{10}$ correct to 13 places of decimals), approximation of π by $\sqrt{10}$, some concepts of infinity, laws of indices and the formulae for permutations and combinations.

(iii) The *Bakhshali* manuscript, found at the village Bakshali near Peshawar in 1081 AD, is believed to belong to the fourth century AD. It contains problems on fractions (method of false position), arithmetic and geometric progressions, solution of indeterminate equations, approximation to surds, solution of quadratic equations and simultaneous linear equations.

(iv) Āryabhata (born 476 AD) gave the algorithm for finding cube roots, sums of squares of cubes of first numbers, approximation of π by 3.1416, mensuration formulae and tables for $\sin \theta$.

(v) Brahmagupta (born 598 AD) solved the indeterminate equation $Nx^2 + 1 = y^2$ in integers, gave the sum

of geometrical progression, determined the right angled triangles and cycle quadrilaterals with rational sides, gave interpolation formulae and the formula for the area of a cycle quadrilateral.

(vi) Mahavircharya wrote his famous work, *Ganita Sara Sanghara*, in 850 AD. It contains problems on arithmetic, on quadratic equations, on the general formula for cube root, on unit fractions, on construction of cycle quadrilaterals of any given area and results for the ellipse.

(vii) Bhaskaracharya (born 1114 AD) wrote *Siddhanta Shiromani* at the age of thirty six, wherein he systematised mathematics known at that time. It gave formulae for surface area of a sphere and its volume, volume of frustums of a pyramid, problems on permutations and combinations, problems on quadratic equations and indeterminate equations. Bhaskara also gave some results which show that he could almost have discovered the calculus.

(viii) After Bhaskara, new mathematics was not searched except in Kerala, where a great deal of mathematical work was done. *Ganit Kaumudi* by Narayan Pandita of the fourteenth century, and the work of Madhava, are pioneers. They are followed by *Tantra Sangraha* and *Āryabhatiyam of Sakara Varman, Karana Paddhati, Drigganita* and *Kariya Kranakara*. In these works, series for sin x, cos x, $\tan^{-1} x$, etc., are given at least two hundred years before they were found in Europe. The result,

$$\lim_{n \to \infty} \frac{1^p + 2^p + \ldots + (n-1)^p}{n^{p+1}} = \frac{1}{p+1}$$

is also given. *Karana Paddhati* gives the value of π correct to 10 places of decimals, and *Sadratanmala* gives it correct to 17 places. Nilkantha also clearly stated, in about 1500 AD, that the ratio of the circum-

ference to the diameter can never be expressed as the ratio of two integers.

Hindu mathematicians in ancient India were highly influenced by the Greeks at an early stage, and by the Chinese at a later stage, but they also contributed a lot to the development and research of mathematics. The nature of mathematics is two-fold. On the one side, it deals with quantitative relationships between material objects, and thus, finds applications in business, economics, sciences, etc. On the other, it postulates and seeks to carry these theorems to their logical conclusions, without any practical use. From this point of view, mathematics can be considered as philosophy.

The decimal system

The first of the main aspects of mathematics had its origin in the counting, weighing and measuring needed in primitive trade. This seems to have included the invention of the numerals that developed in the Hindu-Arabic system, and the zero, which made the place value idea possible. At the time of introduction, the numerals and the zero were not rated as being of particular importance. Other systems of numerals were in vogue, and these were apparently quite as valuable as the ones from which our familiar symbols were derived.

The most important fundamental contribution of ancient India to the progress of civilizations, is the invention of what is called the decimal system of numeration, including the invention of the number zero. The characteristic feature of the system is the usage of nine digits and a symbol for zero to denote all integral numbers, by assigning place value to the digits. This was known as the Decimal System.

A well-known archeologist, Dr. S.R. Rao, has

professed a new theory, that it was the Harappans who gave to the world the decimal system of measurement and the use of gold as a medium of exchange. Dr. Rao is of the view that Harappans produced gold discs which were made in a particular ratio of weight. The unit was known as *gunja*, conforming to the *Suvarna Mashaka*, referred to in the *Arthasastra*. The gold discs of various sizes found at Lothal, and hitherto mistaken for pendants, weighed exactly 50, 100, 2500, 2750, 2800, 2900, 3000, 3250 milligrams, thus indicating a decimal system.

Dr. Rao claimed that it was the Harappans who introduced the decimal system for linear measurement and for weighing delicate objects like gold and precious stones. The divisions marked on the Lothal ivory scale in decimal system were the smallest known in the ancient world, for each division measured 1.7 mm. However, the scales found at Harappa, Mohenjodaro and Lothal were all differently marked, but were interrelated. Dr. Rao said, "The vast structures, such as the dockyard at Lothal, were well proportioned in terms of the Lothal linear scale. The Harappan measurement system was continued in later days. The ten divisions of the Lothal scale were equal to one angula mentioned in the *arthasastra*." According to his view, Harappans achieved "great accuracy" in the measurement of fundamental units of length as well as time. The smallest unit of length was 1.7 mm and that of mass was 50 mg.

There is also a school of thought, which believes that, in spite of lack of direct evidence of a great deal of mathematics, the high level of Mohenjodaro and Harappa civilizations with well-planned cities, pucca houses with ventilation and a highly developed irrigation system could not have been possible, without a corresponding strong development of mathematics.

Numerals and zero symbol

The profundity of the invention of place numerals is understood only when one realizes the difficulty of making progress in arithmetic using other systems, such as the Greek system of numerals I, II, X, C, etc. The Greek method of representing numbers by geometrical segments, and the slow progress of mathematics in the West before the advent of the Indian system in the West, are glaring examples of the handicaps which the people were realizing, because they were not acquainted with the Indian decimal system. The importance of the decimal system of numeration can best be appreciated in the words of Laplace (1749–1827),one of the greatest mathematicians of all time. He wrote, "the idea of expressing all quantities by nine figures (or digits), whereby is imparted to them both an absolute value and one by position, is so simple that this very simplicity is the reason for our not being sufficiently aware how much admiration it deserves".

The invention of a symbol for zero came some time after the idea of a zero became common property. Thus, the man who devised a symbol for zero did a great service to civilization. Neither the names of these persons nor the period when they lived are known with identity, as it was not until the numerals had not been in use for centuries. It was also because the history of mathematics in India reached the stage of preserving the names of important contributors, much later.

In an article in the American Mathematical Monthly, professor G.B. Holstead shows that the zero existed in India at the time of Pingala's work Chhandas-Sutra a work on prosody before 200 BC. He remarks, the importance of the creation of the zero mark can never be exaggerated. This going to airy

nothing, habitation and a name, a picture, symbol, but helpful power, is the characteristic of Hindu race whence it sprang. It is like Coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of the intelligence and power. In fact the zero symbol was used in metric by Pingala (before 200 BC) in his Chemdas-Sutra.

1.5 Development of Mathematics in Various Countries

We illustrate the periods during which mathematics developed in various countries in the following figure:

Babylonia

There are 300 mathematical clay tables containing mathematical results and lists of mathematical problems. The first group of tables belongs to about 2100 BC, the second to about 1600 BC and the third group belongs to a period between 600 BC and 300 BC. Mathematics obviously developed prior to 2000 BC in Babylonia.

Egypt

The great pyramid of Gizah, which was erected about 2900 BC, shows mathematical talent of a high order. The Moscow Papyrus, containing 25 problems, belongs to about 2850 BC and the Rhind Papyrus, containing 85 problems, belongs to about 1600 BC.

India

Mohenjodaro excavations of about 5000 years ago show highly developed mathematics. The *Sulvasutras* belong to the period between 800-500 BC. *Jain*

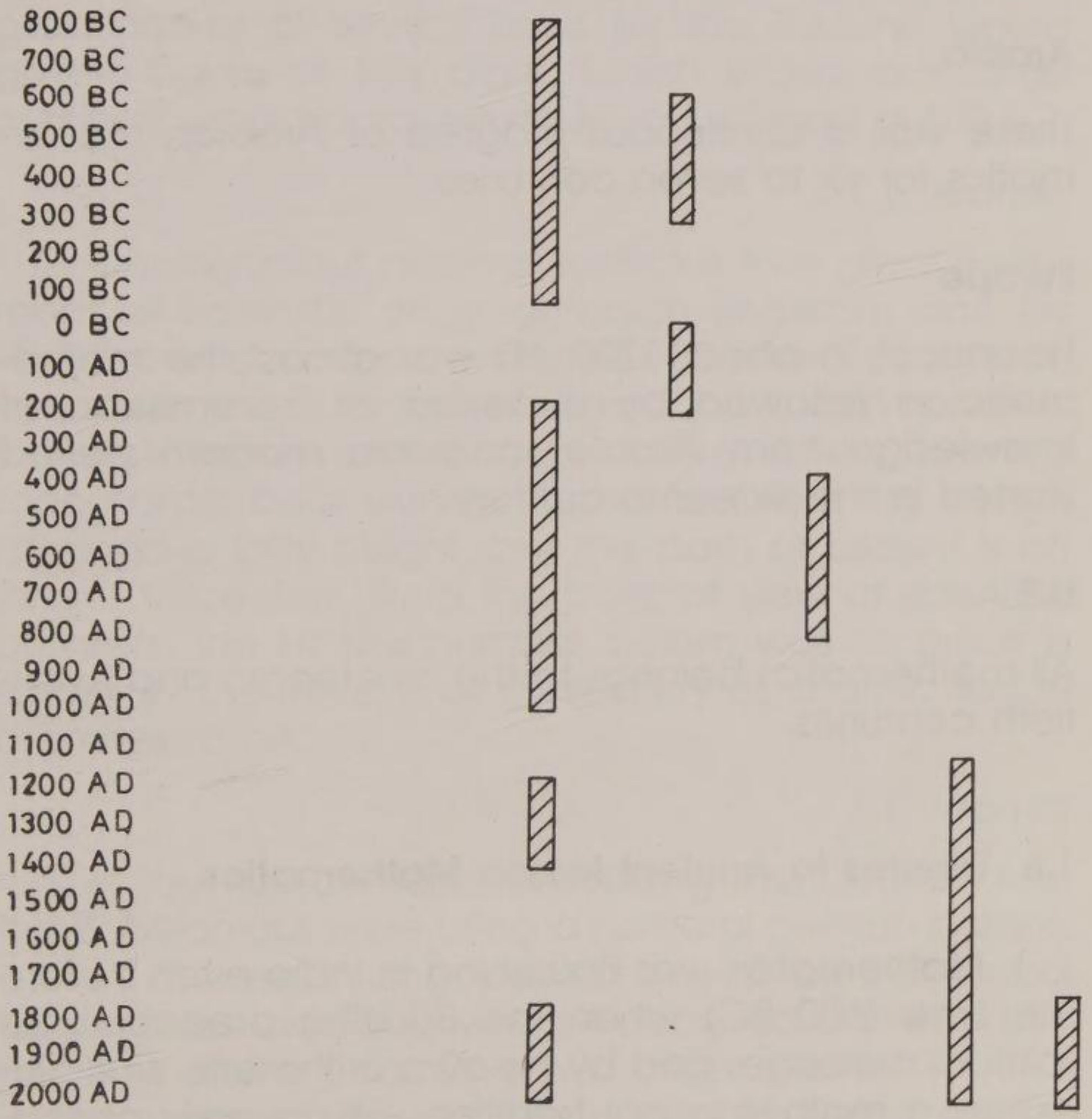


Fig. 2

Shastras to about 500-100 BC and then there is the great mathematical renaissance of Āryabhata, Brahmagupta, Mahaviracharya, Bhaskara from the fifth to the twelfth centuries AD.

Greece

The first period produced Pythagoras, Euclid, Archimedes and Appollonius, and the second period produced Diophantus and Pappus.

Arabia

There was a continuous progress of Arabian mathematics for six to seven centuries.

Europe

Fibonacci, in about 1200 AD was almost the mathematician followed by a period of transmission of knowledge from Arabia, and the modern period started in the sixteenth century.

U.S.A.

All mathematics belongs to the nineteenth and twentieth centuries.

1.6 Tributes to Ancient Indian Mathematics

1. Mathematics was flourishing in India even before the time (600 BC) when the Buddha preached his spiritual message, and by his own arithmetic work, initiated a mathematical tradition. At an early period, the Hindus were skilful calculators, even where large numbers were involved. When the Buddha was matched against Arjuna, a great mathematician of the day, and asked to demonstrate his scientific knowledge, he proceeded to describe the number of 'Primary atoms' which, placed end to end, would form a line equal in length to the ancient Hindu equivalent of a mile. He stated "seven primary atoms

make a very minute grain of dust, seven of these make a minute grain of dust, seven of these make a grain of dust whirled up by the wind. Thus, he continued, until he reached the length of a mile. The multiplications of all sevens listed by the Buddha would give a figure of fifty digits, which shows that large numbers were no novelty to Hindu science in 600 BC.

E.E. Kramer

2. Our ingenious number system is that great instrument of scientific progress which Bhaskara and his Hindu predecessors developed. Bhaskara's performance was the outcome of more than fifteen hundred years of Hindu mathematical culture. A few scientific manuscripts bear witness to the fact that the Hindus attained a lofty height, but the path of ascent is no longer traceable. From the point of view of scientific progress, the Hindus number system was as great a factor in the history of civilization as atomic fusion promises to be.

E.E. Kramer

3. In the two centuries preceding the Christian era, the Babylonians were using a numeral position system, with 60 as base, in an imperfect way, since the symbol for zero was used only when it occurred in the middle and not at the end. Only two other ancient systems were systematically using the symbol for zero in denoting numbers. That of the Mayas (Central America), which appears to have been in use at the beginning of the Christian era and our decimal system in use today which through Arab intermediaries is derived from Hindu mathematics, in which its use is attested from the first century of our era. It is necessary to note that the concept of zero as a number (not as a symbol of separation in decimal notation) and its introduction in calculation is reckoned among the original

contributions of the Hindus.... The Hindu mathematician Bhaskara (12th century) knew the general formula for the number of combinations of *n* things taken *p* at a time.

N. Bourbaki

4. Although the Greeks had the idea of a real number, they took the number to be positive. They considered only numbers which have a concrete (sensible) representation and excluded others, e.g., zero and negative numbers, which make their appearance in the (high) middle ages in Hindu mathematics.... The denial of the Greeks to consider these as numbers is attested by such names for these numbers, as, false, absurd, impossible, imaginary,.... In this regard the Hindus were already conscious of the interpretation which negative numbers must receive.... (For example, as a debt in commercial transactions). In the subsequent centuries, this knowledge was diffused by the intermediary Arabs, who combined Greek and Hindu mathematics and realised that a number may have interpretations other than geometric or kinematic. This developed, progressively, the algebraic notation, the only notable progress of algebra during the end of the middle ages.

N. Bourbaki

5. The study of the solution in integers of indeterminate equations commenced (not with the Greeks or Diophantus) only with the Chinese and Hindu mathematics of the high middle age.... The Chinese who came across such equations in the formation of the calender, etc., had a rule for solving simultaneous linear equations (without doubt during the period from the fourth to the seventh century AD). Concerning the Hindus whose mathematics knew its full brilliance from the fifth to the thirteenth century AD, they not only

treated methodically, by using the Euclidean-algorithm process, the systems of (Diophantine) equations in any number of variables, but they were the first to begin the study and the solution of second degree equations, which are certain particular cases of the "Fermat-Equation" $Nx^2 + 1 = y^2$.

N. Bourbaki

6. Most of the ancient civilizations (Egypt-Babylonia-India-China) appear to have independently obtained the enunciation at least covering certain particular cases of "The Theorem of Pythagoras" (Note this parenthesis indicates that the name is a misnomer) and the Hindus certainly have had the principles of the demonstration of the theorem entirely distinct from the two proofs in Euclid.

N. Bourbaki

7. The Greeks (Aristarchus, Hipparkus, Ptolemy) were the first to construct the "Trigonometrical Tables" which strictly are the "Tables of chords" to the table of the function: $\theta \rightarrow 2 \sin (1/2)\theta$. The introduction of the sin which is very convenient to manage, is due to the Hindu mathematician of the middle ages.

N. Bourbaki

8. The importance of the creation of the zero mark can never be exaggerated. This, which gives to airy nothing, not merely a local habitation and a name, a picture, a symbol, but helpful power, is characteristic of the Hindu race from which it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power.

Halstead

9. It is India which gave us the ingenious method of

expressing all numbers by means of ten symbols, each symbol receiving a value of position, as well as an absolute value, a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity, the great ease which it lent to all the computation, puts our arithmetic in the first rank of useful inventions, and we appreciate the grandeur of this achievement the more, when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.

Laplace

10. The problem of numeration was formally solved by the Hindus at some controversial date before 800 AD. The introduction of zero as a symbol, denoting the absence of units of certain powers of ten in a number represented by the Hindu numerals, has been rated as one of the greatest practical inventions of all time.

E.T. Bell

11. What the Greek genius could not attain, i.e., to give the most convenient notation of numbers, to extend the system of numbers, and to abstract irrational numbers from their geometric base—all this was done at the first stage in India, and reached Europe through middle Asia and Arabic countries. India was a cradle of one of the most fundamental concepts of mathematics. The glorious history is a guarantee of the bright future of science.

A.D. Alexandrov

12. It was during the dark ages, while mathematics in Europe and Egypt stagnated, that the Hindus developed algebra. They invented the numbers we call Arabic, but their most important contribution was the

positional notation. They also started using special symbols for unknown quantities.

New World of Mathematics

13. India is the motherland of our race; mother through the Arabs of much of our mathematics; mother through Buddha of the ideals embedded in Christianity, mother through the village community of self-government and democracy. Mother India, in many ways, is mother of us all.

Will Durant

1.7 Concluding Remarks

(i) While we may be flattered by the well-deserved tributes to ancient Indian mathematics, we can not shut our eyes to some of the criticism given below:

> "There are many differences between Greek and Hindu mathematics. In the first place, the Hindus who worked in mathematics regarded themselves, primarily, as astronomers, and thus, Hindu mathematics remained largely a handmaiden to astronomy, where Greek mathematics attained an independent existence and was studied for its own sake. Also, due to the caste system in India, mathematics in India was cultivated almost entirely by the priests. In Greece, mathematics was open to anyone who wanted to study the subject. The Hindus were accomplished computers, but mediocre in geometry while the Greeks excelled in geometry, but cared little for computational work. Even Hindu trigonometry was arithmetical in nature. The Hindus wrote in verse and often clothed their works in mystic language, while the Greeks strove for clarity and logic in

presentation. Hindu mathematics is largely empirical, proofs or derivations are seldom offered. An outstanding characteristic of Greek mathematics is its insistence on rigorous demonstration. Hindu mathematics is of very uneven quality, good and poor mathematics appearing side by side. The Greeks, however, distinguished between good and bad mathematics (H. Eves, History of Mathematics, p. 189)."

(ii) The achievements of Hindu astronomy are impressive, but these became subservient to some extent to astrology. After the invention of the telescope, western astronomy moved rapidly forward. However, most of our calculations are still based on *Surya Siddhanta*. We have to update our ancient calculation algorithms in the light of modern developments in observational and theoretical astronomy.

(iii) Indian astronomy did not get as much recognition abroad as it should have, partly because of prejudices of foreigners, and partly because of the vagueness of its predictions and the lack of sufficient efforts to explain lucidly. A massive research effort, coordinated by the scientific community and backed by the government, is required if Hindu astronomy is to get its due place.

(iv) Efforts have also to be made to study the development of mathematics and astronomy in ancient Babylonia, Egypt, China, Arabia, etc., to find out the interaction between India and other civilizations.

(v) Readers may note that we have not included sixteen *sutras* and thirteen *sutras* ascribed by Swami Krishna Tirthaji Maharaj to the *Atharvaveda*, and from which, according to him, answers to every mathematical question can be obtained in one line only. This is because there is absolutely no evidence of these *Sutras* being part of the *Vedas*, or their having

influence later development of mathematics in India or in the world.

In a recent book, *Modern Civilization and Living Ancient Mathematics*, the author paid the following tribute to Brahmagupta:

> "Brahmagupta, more than any other single mathematician of the world, helped to accelerate the growth of our economy, commerce, mathematics, technology and most of the aspects of our civilization. Brahmagupta is the most indispensable mathematician of the world, without whose work the progress of civilization would have been severely affected. Brahmagupta contributed more to human civilization than any other mathematician."

According to him, Brahmagupta was the greatest mathematician of all time, even surpassing the three mathematicians who are considered the greatest, namely, Archimedes, Newton and Gauss. However, it may be noted that Brahmagupta himself drew inspiration from the pioneer Āryabhata, and held him in the highest esteem. We may even say that if there have been no Āryabhata, there would have been no Brahmagupta.

CHAPTER 2

LIFE HISTORY OF ĀRYABHATA

2.1 Introduction

April 19, 1975 was a gala day in the history of India. On this day, India launched a satellite into space and drew a lot of praise from every corner of the world. Every Indian feels proud of this achievement, which put India in the 11th position in space research. The Indian satellite was designed and built in Peenya, near Bangalore in Karnataka, by the scientists of the Indian Space Research Organization (ISRO). It took more than twenty six months to complete the first model and Rs 50 million were spent on it.

The satellite was named "Āryabhata." Why? What was the significance behind this name? These are the questions which arise in our minds. Those who have some interest in the achievements of our ancestors in the field of mathematics, know about Āryabhata. He was one of the four most important mathematicians of ancient India, namely Āryabhata, Brahmagupta, Mahavira and Bhaskara. The work of these mathematicians is characterized by a strange mixture of things and is of little account with those who are extraordinary genius and brilliant. It was very difficult for a scientist and necromancer to maintain a high standard in all of their work. This phenomena was not only

prevailing in India, but in other foreign countries also. It was true in the case of John Napier, the inventor of logarithm, who was popularly and profoundly supposed to deal in black magic.

The first of the great Hindu mathematicians, Āryabhata, lived in or near the present city of Patna on the Ganges, to the east of Banaras. Prof. D.E. Smith [28] has written the following introductory remark in his book, *History of Mathematics*:

> "Having paid homage to Brahma, to Earth, to the Moon, to Mercury, to Venus, to the Sun, to Mars, to Jupiter, to Saturn, and to the constellations, Āryabhata, in the city of flowers, sets forth the science venerable."

Āryabhata was one of the greatest mathematicians and astronomers that India ever produced. He made magnificent contributions in mathematics and played a significant role in the evolution of Indian astronomy. History establishes the fact that India was very advanced in astronomy during the period of Āryabhata, and provided leadership to the entire world in this field in the earliest works available, namely, the *Vedas*. Well-developed systems of metaphysical, social, religious philosophy and other sciences, which helped us in making up the present civilization, are to be found.

The Indian Government must be praised for paying tributes Āryabhata, the father of Indian astronomy, by naming India's first satellite after him. The Government also made a thoughtful gesture by celebrating the 1500th birth anniversary of Āryabhata on 2nd November, 1976. It was celebrated by the Indian National Science Academy from 2nd to 4th November, 1976 in New Delhi, and our former Prime Minister, the late Mrs Indira Gandhi, inaugurated the celebrations. In her inaugural address she remarked:

"Āryabhata was a remarkable man of clear thinking, and rational and courageous enough to express his views in contradiction of accepted theories. He was a great figure of Indian history and a bright star in the firmament of world science. The *Upanishads*, the *Mahabharata* with the incomparable *Gita*, the *Ramayana*, the teachings of Buddha, the idea of *ahimsa*, the concept of zero, the *natya shastra*, the sculptural skills of the Indian hands, the experiments of Mahatma Gandhi and the theory of non-alignment—all these are on the concurrent list of India and humanity. To this list belonged Āryabhata and his insights."

The most significant fact about Āryabhata is that he was instrumental in stirring up a lot of scientific activities with some of his original and unorthodox ideas. This is evident from a number of commentaries on his works in later centuries by several scholars, some admiring, but some of them criticizing him. This is understandable because he was far ahead of his time.

Āryabhata's contributions in the field of mathematics are enormous. No progress worth mentioning was made in mathematics till 300 BC, and our knowledge of the history of Indian mathematics prior to 499 AD is very imperfect and undependable. The only written document belonging to the period prior to 499 AD "Surya Pragati" (200 BC) of the Jains and it depends mainly upon the principles of *Veda Jyotish*. Barring this knowledge, we are wholly in the dark as regards the mathematicians who lived prior to Āryabhata, and their works. It was Āryabhata who gave a systematic representation to the arithmetic known before his time. In fact, the *Āryabhatiya* of Āryabhata is the only written document accepted by most of the scholars dealing with the fundamental principles of mathematics and astronomy.

Āryabhata wrote on arithmetic and dealt with arithmetic and geometric progressions, quadratic equations and indeterminate equations. His work in mensuration was relatively crude and it somewhat resembled with that of the Egyptians. Āryabhata's contributions are significant in the field of astronomy.

He invited the attention of the entire world by stating that the earth moves around the sun and the stars are stationary. This revelation about the earth and the stars given by Āryabhata was much beyond the imagination of scholars even a thousand years after Āryabhata. They were unable to understand the discoveries made by Āryabhata, which was why he became the subject of criticism.

"अनुलोमगति नौर्स्थ पश्यत्यचलं विलोमगं यद्धत्।
अचलानि भानि तद्धत समपश्चिमगानि लंकायाम् ॥"

The above lines translate as, "Just as a man sitting in a boat going forward sees a stationary object moving backward, so at Lanka (a hypothetical place in 0 latitude and 0 longitude) do the stationary stars appear to move westwards."

Parthdak Swami (860 AD) and Copernicus (1600 AD) supported Āryabhata and his principles of rotation of the earth around the Sun and the stationary character of stars. It is well known that Galileo was crucified only because he said, "The Earth is rotating around the sun," which was against the religious beliefs of the Christians at that time. These facts clearly prove that India was highly advanced in astronomy, the whole credit for which goes to Āryabhata. It can be rightly said that he was the first Indian scholar to provide the scientific formulation of Indian astronomy.

There are some Indian authors who severely criticized Āryabhata. We may mention the name of Brahmagupta (598 AD) who was one of the greatest

mathematicians and astronomers of ancient India and was called 'Ganaka Cakra–Curamani' by Bhaskara II. He was a bitter critic of Āryabhata during his young age. Everywhere he tried to find faults in the principles and scientific thoughts of Āryabhata. However, he was accused by Alberuni who had been full of admiration of Āryabhata and his work, for undue harshness and animosity against Aryabhata and expressed his feelings as follows:

> "He is rude enough to compare Āryabhata to a worm, eating the root, by chance describes certain characters in it, without understanding them and without intending to draw them."

However, Brahmagupta does not criticise Āryabhata as mathematician. He no where mentions Āryabhata in his mathematical sections although his subject matter is, in many cases, identical with that of *Ganitpada* of the Āryabhatiyam. Even Bhaskara II (1114 AD), the greatest astronomer and mathematician of ancient India, never criticized Āryabhata and quoted some of mathematical and astronomical principles of Āryabhata in his work.

Suryadasa (1541 AD), a commentator on the works of Bhaskara II also pays due regards to Āryabhata and Brahmagupta as ancient mathematicians of repute. Most of the later Indian and foreign authors regarded Āryabhata as a great authority and quoted in their works. G. Thibant speaks of his greatness in the following words:

> "So great was his position that whoever differed him, was the subject of ridicule, even by men who could never aspire to be the author of a treatise on scientific Indian astronomy. Brahmagupta no more than one instance, rightly finds mistakes in Āryabhata's method. So great was his

fame that his mistakes even were taken as oracles and he was generally regarded as the oldest of scientific Hindu astronomers."

2.2 Birth

Aryabhata is also called as Aryabhata I. This means that there must have been more than one Āryabhata. A fundamental question, which has still not been definitely answered, is the one about the number of mathematicians who lived by the name Āryabhata. Evidences show that there was another Āryabhata who was born in about 950 AD. The Muslim historian, Al-Biruni, has mentioned in his book in 1030 AD that there were two Āryabhatas, the elder was known as Āryabhata I and the other one as Āryabhata II. Further, he says that the latter has followed the doctrines of the former. In fact Al-Biruni was not aware of Āryabhata who wrote *Maha-Siddhanta* and born in about 950 AD and so it can be concluded from his history that there was another Āryabhata prior to the famous Āryabhata. No doubt, Al-Biruni's history contains many mistakes and inconsistencies, but after careful examination of his references to the works of two Āryabhatas, one can conclude that all of his references have been taken from the *Āryabhatiya* alone. However, there are some facts in support of Al-Biruni's statement. In the preface to his book, Aryabhata of 950 AD write:

"वृद्धार्य भटप्रोक्त सिद्धांताद्यं महाकालात्।
पाठैर्गतं उच्छेदं विशेषितं तन्मया स्वोक्त्या ॥"

In this slóka he says "the principles of astronomy, as given by the elder Āryabhata are odd and full of errors, so I am giving in my own language." This

statement of Āryabhata of 950 AD refers not to the Āryabhata of Kusumpura, but to another Āryabhata who might have lived earlier, as there is absolutely no similarity between the principles of the *Āryabhatiya* and the *Arya-Siddhanta*. Thus, it seems that there were three Āryabhatas. This is also confirmed by Brahmagupta (628 AD) in his writings. At one place, he has criticized Āryabhata in his work *Brahma Sphuta Siddhanta* written in 628 AD, while at another place in his later work, the *Khandakadyaka*, he refers to Āryabhata with great reverence. The Āryabhata referred to in the *Khandakadyaka* is not the Āryabhata of Kusumpura, since the positions of the planets etc. in the *Khandakadyaka* and in the *Āryabhatiya* are very different and lead to this conclusion. It is also said that Brahmagupta became repentant in his old age and has therefore given encomiums to Āryabhata in the book written in his old age. In view of conflicting statements and facts, a final conclusion about the identity of Āryabhata has still not been reached. It appears that there were two Āryabhata's before Brahmagupta. However, let us not indulge in this controversy over the number of Āryabhatas. In this book, we will deal with Āryabhata I, the author of the *Āryabhatiya*, although we will always refer to him as Āryabhata.

There are different opinions about Āryabhata's the exact date of birth. However, the year of his birth is known with precision. Āryabhata himself has mentioned his date of birth in his famous work the *Āryabhatiya*. The relevant *slóka* is

"षष्टयब्दाना षष्टिर्यदा व्यतीतास्त्रथश्च युगपादा:।
त्र्यधिका विशतिरब्दास्तदेह मम जन्मनोऽतीता: ॥"

According to this slóka "Now when sixty yugas of sixty years and three quarter yugas also have elapsed, twenty increased by three years have

elapsed since my birth," Sixty yugas of sixty years = 3600. One of four quarter yugas viz, Satya, Treta, Dvapara and Kaliyuga, first three yugas have passed. According to Hindu astronomers the fourth yuga commenced on February 18, 3102 or 3179 years before saka era. In this way Āryabhata was 23 years old when 3600 years of the present Kaliyuga had elapsed i.e in 3600-3179 or 421 of the Saka era which is equivalent to 499 of the christian era. Consequently, he was born in 476 AD and in the same year Budha Gupta, the king of the Gupta period (320 AD to 740 AD)took over the reigns of the kingdom of Pataliputra and ruled till 494 AD.

Gupta period (320 AD to 740 AD) is called the Golden Age of India. During this period there was an outburst of intellectual and scientific activities. The people of this period possessed a desire to learn new things in science, literature and other spheres. R.C. Majumdar rightly says:

"It was during this period that Indian intellect reached its high watermark in most branches of Arts, Science and Literature. Indian culture and civilization reached a unique stage of development which left its deep impression upon succeeding ages." He further adds that it is this period that produced the best authors in almost all branches of literature including even sciences like astronomy and mathematics.

There are two schools of thought about Āryabhata's exact date of birth. One school believes his date of birth to be March 21, 476 AD, while the other believes it to be April 13, 476 AD. According to the first school of thoughts, 3600 years of the *Kali* era came to an end on Sunday, March 21, 499 AD, at mean noon at Ujjayani, at the time of mean sun's entrance into the sign Aries (*Madhyama Mesa Sanskranti*). Therefore, it can be said that Āryabhata was born at *Mesa-Sanskranti* on March 21, 476 AD.

Table 1
Mean positions of planets at *Kali* 3600 elapsed

Planet	*Ārya-bhatiya*	*Āryabhata-Siddhanta*	Ptoleming	Moderns
Sun	0°0'0"	0°0'0"	357°8'16"	359°42'5"
Moon	280°48'0"	280°48'0"	278°24'58"	280°24'52"
Moon Apogee	35°42'0"	35°42'0"	32°43'42"	35°24'38"
Moon's asc. node	352°12'0"	352°12'0"	349°25'33"	352°2'26"
Mars	7°12'0"	7°12'0"	4°20'12"	6°52'45"
Mercury	186°0'0"	180°0'0"	178°0'27"	183°9'51"
Jupiter	187°12'0"	186°0'0"	185°20'55"	187°10'47"
Venus	356°24'0"	356°24'0"	351°4'15"	356°7'51"
Saturn	49°12'0"	49°12'0"	45°55'39"	48°21'13"

It is not clear why Āryabhata I mentioned the end of the *Kali* year 3600, which happened to occur on Sunday, March 21, 499 AD, at mean noon at Ujjayini. It is also not clear whether this time denotes his date of birth or the time of composition of the *Āryabhatiya*. According to the commentators on the *Āryabhatiya*, the object of specifying the *Kali* year 3600 was to show that at that time, the precision of the equinoxes amounted to zero, and the mean positions of the planets, obtained from the astronomical parameters given in the *Gitika-pada*, did not require any correction. In the opinion of the commentators Suryadeva (b, 1191 AD), Paramesvara (1431 AD) and Nilakantha, this was the time of composition of the *Āryabhatiya*. This view was supported by K. Sambasiva Sastri, W.E. Clark and Baladeva Mishra. While P.C. Sengupta (1927 AD) once entertained this view, he later discarded it.

The Kerala astronomer Haridatta (C. 683 AD), the author of the so-called *Sakabda* correction, has interpreted the *slóka* on page 34 in a different way, namely, "when sixty times sixty years and three quarter-*yugas* had elapsed (of the current *yuga*), twenty three of my age had passed since then." No commentator on the *Āryabhatiya*, not even of Kerala, has interpreted the above passage in this way. T.S. Kuppanna Sastri has called it a wrong interpretation. Another Kerala astronomer who was the author of *Drkkarana* (1603 AD), an astronomical manual in Malayalam, has stated that Āryabhata I was born in 499 AD and wrote the *Āryabhatiya* twenty three years later, in 522 AD. This interpretation was also turned down by T.S. Kuppanna Sastri and others.

It is worth mentioning that the translation of the *slóka* in question, as given earlier, is in agreement with the interpretation of the commentators, which is also in conformity with what Āryabhata himself told his pupils while teaching the subject. This was the view of Bhaskara I also. This duality of interpretation of the above *slóka* has given rise to two epochs, called *Bhatabda*, associated with Āryabhata I, viz., *Saka* 421 = 499 AD and *saka* 444 = 522 AD, and to two *bija* corrections, one taking the beginning of *saka* 421, and the other the beginning of *saka* 444, as the zero point.

Since at the end of the *Kali* year 3600, the precession of the equinoxes amounted to zero, the amount of the precession of the equinoxes 23 years before, at the time Āryabhata's birth was negligible, therefore, his birth may be taken to have occurred at *niryana-mesa-sankranti* or *sayanamesa-sankranti*. On the other hand, the Bihar Research Society, Patna, which belongs to the second school of thought, celebrates the birth anniversary of Āryabhata on April 13 i.e, on

the *nirayana-mesa-sankranti*, the day on which the sun enters into the *nirayana* sign.

2.3 Kusumpura of Āryabhata

"सिद्धान्तपंचकविद्यावपि दृग्विरुद्ध
मौढयोपरागमुखखेचर चार कतृत्तौ।
सूर्यः स्वयं कुसुमपुर्यभवत् कलौ तु
भूगोलवित् कुलप आर्यभटाभिधान ॥"

This very first *slóka* of the *Āryabhatiya* confirms the fact that Āryabhata was born in Kusumpura, and wrote his famous work, *Āryabhatiya*, in 499 AD, at the age of 23 years. Besides the other contributions of Āryabhata, this immortal work is enough to put him among the top ranking astronomers and mathematicians of the world.

Āryabhata and his birth place, Kusumpura, are immortalized due to the *Āryabhatiya*. Various commentators, like Paramesvara (1360-1455 AD), Raghunatharaya (1597 AD) and the Persion scholar Al-Biruni (973 AD), have regarded Kusumpura as the place of birth of Āryabhata and their opinion is mainly based upon this very first *slóka* of the *Āryabhatiya*. Apart from these sources, Bhaskara I (629 AD) has identified Kusumpura as Pataliputra or modern Patna, and pin-pointed it as the place of birth of Aryabhata.

There is some controversy about Patna being identified with Kusumpura. Prof. D.E. Smith [28] in his book *History of Mathematics*, has critically discussed this aspect, and has quoted several authorities who believe that Kusumpura should not be identified with modern Patna, rather a place near modern Patna. The place Kusumpura is believed to have been not far from the present Patna. No doubt, Āryabhata belonged to the

Kusumpura school, but it seems probably that he was a native of Kerala, in the extreme south of India. Even now, the calender based upon the system given in the *Āryabhatiya* is followed, to some extent, in Kerala. It is a historical fact that Patna was called by different names. For example, Mohammedans Azimabad called it by the name Patna, while ancient Buddhists called it Patliputra and the Syrian ambassador Magasthenes called it Palibothra.

Because of geographical proximity and historical facts, Āryabhata is often said to have been born at Pataliputra. It is believed that the city of Pataliputra, was founded by Putraka, the knight of the magic cup and staff and slippers, who married the princess Patli. By the beginning of the fifth century, and nearly a century before the birth of Āryabhata, it had lost some of its ancient prestige. One Chinese traveller, Fahien describes the ruins of the royal palace which Ashoka built on this place as well as though he speaks of the remarkable hospitals and other institutions still to be found there. Katha Sarita Sagar also supported this view which, as K.M. Panikkar says, "Embodying the views of Ganadhya's time (1st Century BC) consistly speaks of Pataliputra as the home of culture, learning and fine arts and the queen of the cities of the world known as 'the city of flowers,' its prestige in imperial times was something unique and it seems to have been completely cosmopolitan in its character. Undoubtedly it dominated the intellectual life of India for many centuries as we have continuous allusions to students flocking to Patliputra for study."

Further, S. Pillai views that "Āryabhata was born in the village of Asuakam in Kusumpura near modern Patna in the year 3577 of the Kaliyuga, answering to 476 AD." Thus, in view of the evidences available, it is reasonable to assume that Āryabhata was a resident of Kusumpura identical with Patna, the capital of

Bihar. Considering Āryabhata as a resident of Patna, the Bihar Research Society, Patna celebrates his birth anniversary on 13th of April, every year.

Āryabhata's works remained a little unknown among Hindu scholars of centuries immediately following his period, it was probably because Āryabhata lived so far from Ujjain, the ancient centre of mathematics and astronomy.

2.4 Teacher and Devotee of Brahma

There is not so authentic information available to us regarding the life of Āryabhata. Even the *Āryabhatiya* does not throw light on such aspects as his parentage, his educational career or other details of his personal life. From the writings of Bhaskara I. who belonged to his school, it appears that Aryabhata took up the profession of a teacher. He earned a name as a great scholar and teacher of astronomy. Bhaskara I has referred to him as in his writings *Acarya* (Professor) and *Sarva-Sidhhanta guru*. In the *slóka* given in the previous section, Āryabhata has been designated as *Kulpa*, which means *Kulpati* or the Head of the University. It seems that Āryabhata was *Kulpati* of the University of Nalanda, which was one of the best centres of education in the country available at that time.

From the homage which Āryabhata pays to Lord *Brahma* in the opening stanzas of the first and the second chapters of the *Āryabhatiya*, it is sufficiently evident that Āryabhata belonged to the school of astronomy promulgated by *Brahma*. At one place he writes:

"ब्रह्मकुश शिबुधभृगुरविकुज गुरुकोणभगणान्न मस्कृत्य।
आर्यभटस्तिवह निगदति कुसुमपुरेडम्यचितं ज्ञानम् ॥"

Āryabhata acknowledges the grace of *Brahma* on the successful completion of the *Āryabhatia*, in one of the closing stanzas of the same book. This shows how deeply he was devoted to *Brahma*, and this devotion has led some people to suppose that Āryabhata acquired his knowledge of astronomy by performing penance in propitiation of Lord *Brahma*.

In his commentary, the *Maha-Bhaskariya*, on the Āryabhatiya, Bhaskara I writes:

> "This *Ācārya* worshipped Lord *Brahma* by doing severe penance. So, by his grace was revealed to him the true knowledge of the subjects pertaining to the true motion of the planets. It is said that Āryabhata, followed exactly in the footsteps of Vyasa, the son of Parasara, the ornament among men, who by virtue of penance, acquired the knowledge of the subjects beyond the reach of the senses the poetic eye capable of good to others."

Āryabhata's devotion to *Brahma* was indeed of a high order. For, according to his view, the end of learning was the attainment of the supreme Brahma, and this could easily be done through the study of astronomy. In the closing stanzas of the *Dasagitikasutra*, he says:

"दशगीतिका सुत्रमिदं भूग्रह चरितं भपन्जरे ज्ञात्वा।

ग्रहभगणा परिभ्रमणं स याति भित्वा परं ब्रह्म ॥"

Thus, according to him, one who knows these ten slókas which describe the motion of the earth and the planets on the celestial sphere, attains the supreme Brahma, after piercing through the orbit of the planets and the stars. From this we may inter that the perfect knowledge of the universe leads to the path of salvation i.e. the reunion of the Jiva and Brahma.

Āryabhata's predilection for the Brahma school of astronomy might have been inspired by two main considerations. Firstly, the Brahma school was the most ancient school of Hindu astronomy, promulgated by Lord Brahma himself. Secondly, the astronomers of Kusumpura, where Āryabhata lived and wrote his *Āryabhatiya*, were the followers of that school. Bhaskara I writes that the learned people of Kusumpura held the *Svayambhuva-Siddhanta* in the highest esteem, even though the *Poulisa*, the *Romaka*, the *Vashistha* and the *Surya Siddhantas* were also known there.

S. Pillai says, "that Āryabhata married and had a family is evident from the fact that he begot a worthy son in Devarajan, who afterwards became the author of an astronomical work." But there is hardly to corroborate this view. Even it is not possible to say when and where Āryabhata breathed his last. Professor D.E. Smith [28] writes in his book that Āryabhata died in about 550 AD, but to what extent he is correct in his statements can not be said definitely.

2.5 Pupils of Āryabhata

As we have already mentioned that not much information regarding the life of Āryabhata is now available in literature. Even the *Āryabhatiya* does not throw light on such aspects as his parentage, his educational career and other details of his personal life. One can infer, from the writings of Bhaskara I (629 A.D), that Āryabhata took up the profession of a teacher. Bhaskara I mentions the names of Panduranga-swami, Latadeva and Nisenku, among those who learnt astronomy at the feet of Āryabhata. Latadeva is better known than the other two pupils of Āryabhata, and gets special mention due to his contribution to astronomy. He earned a name as a great

scholar and teacher of astronomy. He was also called *Ac̄arya* (learned teacher) and *Sarva-Siddhanta-Guru* (teacher of all systems and astronomy) by Bhaskara I. We learn from the writings of Varahamihira (587 AD) and Sripati (1039 AD) that Latadeva was the author of at least two works on astronomy. In one, the day was the measured from midnight at Lanka (lat. 0° long. 75°.43'E). Varahamihira has also ascribed to Latadeva the authorship of two commentaries, one on the *Romaka-Siddhanta* and the other on the *Poulisa-Siddhanta*. According to the Persian scholar, Al. Biruni (973 AD–1048 AD), Latadeva was the author of a *Surya-Siddhanta*. Brahmagupta (628 AD) and Prthudaka (860 AD) have also made references to *Acarya* Latadeva. Prthudaka has quoted a number of verses from one of Latadeva works. These verses are in *arya* metre and their language and style are similar to the language and style of the *Āryabhatiya*.

It is certain that Bhaskara I was not a direct disciple of Āryabhata, but a very close follower and one of the earliest exponents of the Āryabhata School. He was a contemporary of Brahmagupta and wrote three works on astronomy, viz, the Mahabhaskariya, the Āryabhatiya-Bhasya and Laghu- Bhaskariya. In fact Bhaskara I's available works are the exposition and interpretation of the Āryabhatiya. Dr. K.S. Shukla [25] remarks "In the absence of the works of Bhaskara I, many a passage in the Āryabhatiya of Āryabhata I would have remained obscure to us". There are frequent references to Āryabhata, the Āryabhatiya and the Āryabhata School of astronomy in his works. In this connection T.S. Kuppanna Sastri tells us, "What the Āryabhatiya gives in the form of Sutra like Aryas; the Maha-Bhaskariya gives elaborately, thus forming an exposition of it" (refer [16] p. 28).

Haridatha who lived in the later part of the seventh century is also one of the followers of the school to

which Bhaskara I belongs. He states that Āryabhata's astronomical principles remained in popular use even after Bhaskara I. Lalla (748 AD) also refers to Āryabhata and his school in his work "Sisyadhivrddhida." He uses the word 'TADIYSHISHYA' which means more than the direct pupils of Āryabhata including Bhaskara I. Howèver, Parameswara's assertion that Lalla was a direct disciple of Āryabhata is not correct. On the other hand, Lalla might have been a younger contemporary of Brahmagupta and after Bhaskara I. Lalla himself also did not claim to be in direct pupil of Āryabhata.

Sankaranarayan (869 AD) at the outset of his commentary on Laghu-Bhaskariya gives a list of teachers of this school and mentions Āryabhata, Varahamihira, Bhaskara I, Govindaswami and Haridatta, and bows down with reverence before his giant predecessors, interestly he ignores Brahmagupta and Lalla who preceeded Sankaranarayana, possibly due to the reason that both of them criticized Āryabhata's astronomical principles. Udayadivakaran (1073 AD), a commentator of the Laghu- Bhaskariya, says, "Having bowed to Murar., the Lord of the entire world, and also having paid respectful homage to Acarya Āryabhata, I write an extensive exposition of the smaller work on astronomy composed by Guru Bhaskara." From this we can say that Udayadivakara was also one of the exponents of this school. Brahmadeva who composed his astronomical work, viz., 'Karma Prakash' in the year 1092 AD. tells us frankly that he wrote his work according to the Āryabhata's astronomical principles given in the Āryabhatiya.

CHAPTER 3

THE ĀRYABHATIYA

3.1 Introduction

Āryabhata wrote at least three works on astronomy,—namely, (1) *Āryabhatiya*, (2) *Āryabhata-Siddhanta*, (3) *Surya-Siddhanta-Prakasa*.

The *Āryabhatiya* is well known, while *Āryabhata-Siddhanta* and *Surya-Siddhanta-Prakasa* are known only through references to them in later works. The first two works have been distinguished by Varahamihira on the basis of the criteria adopted in each of them, for the reckoning of the day. He writes that Āryabhata said. "The day begins at midnight at Lanka," Āryabhata also said that "the day begins from sunrise at Lanka." Other differences between the two works of Āryabhata have been noted by Bhaskara I in his *Maha-Bhaskariya*, which was published in 1945 by *Anandashram Sanskrit Granthawali*.

The *Āryabhatiya* deals with both mathematics and astronomy. However, it is very difficult to say that whatever was written in the *Āryabhatiya* was totally unknown before Āryabhata lived. Even Āryabhata did not say this. On the contrary, the 48th and 49th *slókas* of the '*golapada*' of the *Āryabhatiya* read as given:

"क्षितिरवियोगाद् दिनकृदुवीन्द्र्योगात् प्रसाधितश्चेन्दुः।
शशितांराग्रहयोगात् तथैव ताराग्रहाः सर्वे ॥
सदसज्ज्ञान समुद्रात् समुदधृत देवता प्रसादेन।
सज्ज्ञानोतमरत्नं मयानिमग्नं स्वमतिनावा ॥"

Thus Āryabhata himself said that some of his contributions were based upon the knowledge he acquired through many books available to him. He was very much inspired by the available literature and wherever necessary, he corrected and modified the erroneous results. In all probability, Āryabhata wrote two *Āryabhatiyas,* one at the age of 23 years and the other one when he was quite mature. The second edition of the *Āryabhatiya* was completely the revised edition of the first. The *Āryabhatiya* available to us is this second revised edition. It is a beautifully written scientific work.

The *Āryabhatiya* is considered an excellent textbook on astronomy due to brevity and conciseness of expression, superiority of astronomical content and innovations in astronomical methods. It gave birth to a new school of astronomy, the Āryabhata school, whose exponents called themselves disciples of Āryabhata. These disciples of Āryabhata called him '*Bhagwan*' and '*Prabhu*'. They held the teachings of the *Āryabhatiya* in the highest esteem, claiming greater accuracy for them. Bhaskara I was the most illustrious disciple of the Āryabhata school. He wrote a commentary on the *Āryabhatiya,* and two other works on astronomy in elucidation of the teachings of Āryabhata. He earned great fame as a teacher of astronomy and was known as '*Guru*'. The works of Bhaskara II [7] throw a flood of light on the astronomical theories and methods of Āryabhata, and on the earlier followers of Āryabhata. His commentary on the *Āryabhatiya* was recognized as a work of great

scholarship and it was used by most of the subsequent commentators.

The *Āryabhatiya* is considered a popular work on ancient mathematics and is studied till date, at various centres of learning in South India, especially in Kerala. The popularity of the *Āryabhatiya* can be ascertained by the following facts:

1. There is hardly any work dealing with Indian astronomy which does not refer to Āryabhata or quote from the *Āryabhatiya*.
2. There exist a number of commentaries on the *Āryabhatiya*, written in Sanskrit, Telugu, Malayalam and Marathi, by authors hailing from South India.
3. There are many independent astronomical works which are based on the *Āryabhatiya*.
4. Calendrical texts and tables used in South India after the first half of the seventh century AD till the western astronomical tables were introduced, were based on the *Āryabhatiya* or on works based on it.

In North India, Brahmagupta who lived in the seventh century AD at Bhinmal in Rajasthan, made an intensive study of the *Āryabhatiya*. He made use of this work in his *Brahma-Sphuta-Siddhanta*, and a number of passages in that work are strikingly similar to those of the *Āryabhatiya*. Prthudaka, who lived at Kanauj in Uttar Pradesh, wrote his commentary, on *Brahma Sphuta-Siddhanta* in 860 AD. In this commentary, there are a number of passages from the *Āryabhatiya*. Even for finding the volume of a sphere, Prthudaka gives exactly the same rule which is found in the *Āryabhatiya*. Several passages from the *Āryabhatiya* occur in the writings of the Kashmiri scholar, Bhattotpala, written in about 968 AD.

Āryabhata, while writing the *Āryabhatiya*, adopted the sutra method which was popular among the

ancient writers. This has merits and demerits. It is very useful, since it expresses a lot of things in a few words. But it is disadvantageous as it follows a large scope for misunderstanding and misinterpretation. Due to this method the language of the book is terse and obscure, and many times creates confusion in understanding the verses. Therefore, a cautious and careful study is required in order to understand the contents of the book.

3.2 A Collection of Two Compositions

There is some controversy regarding the contents of the *Āryabhatiya*. This point has been thoroughly discussed by Shukla [25] in his book *Āryabhata*. According to him, the *Āryabhatiya* is the collection of two works.

(i) The Dasagitika Sutra

There is some controversy regarding the number of *slókas* in the Dasagitika-Sutra. Shukla [25] has mentioned that it consists *pada* 1 containing 13 *slókas* in total. Out of 13 *slókas*, 10 are in *giti* metre. On the other hand, Satyaprakash [21] mentioned that there are 10 main *slókas* in this book and apart from these 10 *slókas*, there is one extra *slóka*, namely;

"दशगीतिका सुत्रमिदं भूग्रह चरितं भपन्जरे ज्ञात्वा।
ग्रहभगणा परिभ्रमणं स याति भित्वा परं ब्रह्म ॥"

This eleventh *slóka* just tells us that the learning of 10 main *Slókas* of this book leads to the path of salvation.

(ii) The Āryastasta or Āryabhata-Tantra

This consists of the second, third and fourth *padas*, containing, in all, 108 stanzas in *Arya* metre. It is noteworthy that the *Dasagitika-Sutra*, and the *Aryastasata* both begin with an invocatory stanza and end with a concluding stanza in praise of the work, and yet look like two different works.

It seems that the *Dasagitika-Sutra*, which begins with an invocatory stanza and ends with a concluding stanza in praise of its contents, was issued as a separate tract, like the multiplication tables of arithmetic, and was meant for the freshers who used to learn the astronomical parameters by heart, before commencing on the study of astronomy. The *Āryastasata* was meant for those who had learned the *Dasagitika-Sutra* and were qualified for the study of astronomy proper. The north Indian astronomer Brahmagupta (628 AD) has also referred to *pada* 1 of the *Aryabhatiya* as *Dasagitika*, and to the rest of the *Āryabhatiya* as *Āryastasata*.

In addition to the above, there are following evidences which show that *Dasagitika* and *Āryastasata* are two separate books.

a) Both begin with similar invocatory *slókas* and end with the similar concluding *slókas* in praise of the work. Naturally, if these were not two separate books, there would have been only one opening *slóka* and one concluding *slóka*.

b) There are two commentaries on the *Āryabhatiya*, by Yallaya (1480 AD) and Nilkandha (1500 AD), which do not say anything about the *slókas* in *Dasagitika* and only comment on the 108 *slókas* of *Āryastasata*. These commentaries again prove that the *Āryabhatiya* contained only 108 *slókas* and that *Dasagitika* was a separate work of Āryabhata.

c) The commentary of Bhaskara I on the *Āryabhatiya* also makes this distinction, and he clearly writes separate commentaries on the *Dasagitika* and the *Āryastasata* and calls them as *Dasagitika-Viyakhaya* and *Āryabhata-tantra-vyakhaya*, respectively. So, it can be said that the *Āryabhatiya* is the composition of two books, namely, *Dasagitika* and *Aryabhata-tantra*.

Next, we describe briefly the contents of both these books treating them as one unit. There are four parts of the book as given below:

1. *Gitika-Pada*
2. *Ganita-Pada*
3. *Kalkriya-Pada*
4. *Gola-Pada*

1. Gitika-Pada

This part consists of thirteen stanzas, of which ten are in *gitika* metre. It sets forth basic definitions and important astronomical parameters and tables. It gives the definitions of the large units of time (*kalpa, manu* and *yuga*), the circular units (sign, degree and minute) and the linear units (*yojana, hasta* and *angula*).

It states the number of rotations of the earth, the revolutions of the sun, the moon and the planets, etc., in the period of 43,20,000 years. It gives an account of the time and place from which the planets started moving at the beginning of the current *yuga*. Further, it tells about the time that has elapsed since the beginning of the current *yuga*, as well as the time that has elapsed from the beginning of *kalpa* to the beginning of *Kaliyuga*. The positions of the apogees and the ascending nodes of the planets in the time of the author, and the orbits of the sky, the sun, the moon and planets are given in this part of the *Āryabhatiya*. The obliquity of the ecliptic, and the inclination of the moon's orbit to the ecliptic, are started as 24° and

4.5°, respectively. It is believed that the atmosphere above the earth is 12 *yojanas* = 109 miles approximately. This part of the *Āryabhatiya* also contains table of sine differences, which is one of Āryabhata's main contributions to mathematics.

2. Ganita-Pada

This part consists of thirty-three stanzas in all and deals with mathematics. The topics of arithmetic dealt with in this part are geometrical figures and methods of finding their areas, problems on the shadow of the gnomon and mensuration. It has also dealt with series, interest and algebra. The topics of algebra discussed are simple, simultaneous, quadratic and linear indeterminate equations. Further, it gives the arithmetical methods for extracting the square root and the cube root of a number, as well as rules meant for certain specific mathematical problems, including the method of constructing the sine table.

3. Kalkriya-Pada

It contains twenty-five stanzas and deals with the various units of time. It determines the true position of the sun, the moon and the planets. It tells about the divisions of the year (month, day, etc.) and those of the circle. It describes the various kinds of year, month and day, and defines the beginning of the time-cycle, the so-called circle of the sky and chords of hours and days. The motion of the sun, the moon and the planets, by means of eccentric circles and the epicycles, are explained. It also gives the method for computing the true longitudes of the sun, the moon and the planets.

4. Gol-pada

This part consists of fifty stanzas which deal with the motion of the sun, the moon and the planets on the celestial sphere. The various problems of spherical

astronomy are also discussed. It describes the various circles of the celestial sphere, and tells about the method of automatically rotating the sphere once in twenty-four hours. It gives an account of the celestial sphere as seen by those on the equator, the north pole and on the south pole. The rules related to the various problems of spherical astronomy are described, the calculation and graphical representation of the eclipses and the visibility of the planets are also explained in detail.

3.3 Notable Points of the Āryabhatiya

In this section we list the notable features of the *Āryabhatiya* as follows:

1. The alphabetical system of numerical notation

The alphabetical system of numerical notation defined by Āryabhata is different from the so-called *Katapapadi* system, but much more effective in expressing number briefly in verse.

2. Circumference-diameter ratio of a circle

Circumference : diameter = 62832 : 20000 is equivalent to saying that $\pi = 3.1416$. This value of π is correct up to four decimal places, and is more accurate than the value 3.141466 given by the Greek astronomer, Ptolemy.

3. The table of Sine-differences

Āryabhata is probably the first astronomer who gave a table of sine differences. He had also stated geometrical and theoretical methods for constructing sine tables.

4. The astronomical parameters

The astronomical parameters given by Āryabhata in the *Āryabhatiya* are different from those of the other astronomers and are based on his own observations. They are much better than those given by the earlier astronomers. The method applied for their determination has been mentioned in the *Gola-pada* by Āryabhata.

5. Time and division of time

Āryabhata did not believe in the theory of creation and annihilation of the world. He was of the opinion that time was a continuous process without beginning and end. The beginnings of *yuga* and *kalpa*, according to him, have nothing to do with any terrestrial occurrence. They are based purely on astronomical phenomena, depending on the positions of the planets in the sky.

Āryabhata rejected the highly artificial scheme of time division prevailing at that time, and replaced it with the following:

1 day of *Brahma* or *Kalpa* = 14 *manus*

1 *manu* = 72 *yugas*

1 *yuga* = 43,20,000 years.

6. Innovations in planetary computation

We learn from the *Surya-Siddhanta*, summarized by Varahamihira, and from the *Sumati-Mahatantra* of Sumati, that the astronomers before Āryabhata changed the old pattern of correction and in the case of the inferior planets, reduced the number of corrections from five to three. In the case of the superior planets, though the number of corrections

remained the same, but a pre-correction was also mentioned. This innovation was an improvement and yielded more accurate results.

The formula

$$\frac{\text{mandakarna} + \text{sighvakarna}}{2}$$

for finding planetary distances, prescribed in the *Surya-Siddhanta*, was changed by Āryabhata to

$$\frac{\text{Mandakarna} \times \text{Singhrakarna}}{R}$$

7. Theory of the earth's rotation

It was generally believed that the earth was stationary and lay at the centre of the universe, and all heavenly bodies revolved round it. But Āryabhata had a view different from the other astronomers. The view was the earth rotates on its own axis and that the stars are fixed in space. The period of one side real rotation of the Earth, according to Āryabhata, is 23^h, 56^m 4.1^s. This value is quite accurate and near to the modern value, which is 23^h 56^m 4.091^s.

3.4 Commentaries on the Āryabhatiya

(A) Commentaries in Sanskrit

1. Bhaskara I's commentary

Bhaskara's commentary is the earliest commentary on the *Āryabhatiya* out of all commentaries available to us so far. It was written at Valabhi in Kathiawara, in the year 629 AD and gives a comprehensive exposition of the contents of the *Āryabhatiya*. Sankaranarayana (869 AD) writes that those who want to study about Āryabhata should read the

commentary on the *Āryabhatiya*, the *Maha-Bhaskariya*, written by Bhaskara I.

2. Somesvara's commentary

A manuscript of Somesvara's commentary on the *Āryabhatiya* exists in the Bombay University Library, Bombay. The contents of this commentary show that it is a summary of Bhaskara's commentary, as the introductory lines given in this commentary are almost exactly the same as those found in Bhaskara I's commentary. However, Somesvara has given some new examples to illustrate the doctrines in addition to those in the commentary of Bhaskara I. Somesvara's commentary does not throw any light on the life and works of its author. Somesvara belongs to the school of Āryabhata. Up till now only two works written by him are known to us, viz, (i) his commentary on the *Āryabhatiya* and (ii) his commentary on the *Khanda-Khadyaka* of Brahmagupta.

The latter work is not actually available. It has been mentioned by Amaraja (C.1200 AD), who himself wrote a commentary on the *Khanda-Khadyaka*. In addition to Somesvara, Amaraja mentions other commentators of the *Khanda-Khadyaka* whose works he consulted while writing his own. In the second opening stanza of his commentary on the *Khanda-Khandyaka*, Amaraja writes:

> "Having carefully studied the commentaries on the *Khand-Khandyaka* written by Lalla, Utpala and Somesvara, and taking together the meanings given by them, I set forth the same in brief, for the sake of the delight of the readers."

3. Prabhakara's commentary

It is believed that Prabhakara was an earlier commentator than Bhaskara I, as at two places Bhaskara I finds fault with the interpretations given by

Prabhakara. In his commentary, Bhaskara II (1150 AD) refers to Prabhakara a disciple (follower) of Āryabhata. *Acarya* Prabhakara has also been mentioned by Sankaranarayana (869 AD), Udaya-divakara (1073 AD), Suryadeva (1191 AD) and Nilakantha (1500 AD). It is worth mentioning that Suryadeva (1191 AD), in his commentary on the *Laghu-manasa*, refers to Prabhakara as 'Prabhakara-*guru*' and mentions his work the *Prabhakara-ganita*.

4. Suryadeva Yajva's commentary

Suryadeva calls himself Suryadeva Yajva, Suryadeva Somasut and sometimes Suryadeva Diksita. His commentary on the *Āryabhatiya* is known by various names, such as *Āryabhata Prakasa, Bhata-Prakasa, Prakasa, Āryabhata Prakasika, Bhata-Prakasika* and *Prakasika*.

Suryadeva's commentary is an excellent exposition of the *Aryabhatiya* and has been used by Parameśvara (1431 AD) as a source book for writing his commentary on the *Āryabhatiya*.

5. Paramesvara's commentary

Paramesvara's commentary is a brief but excellent exposition of the *Āryabhatiya*. It was edited by H.Kern and printed at Leiden (Holland) in 1874 AD. It was reprinted in 1906 AD by Udaya Narain Singh alongwith his Hindi translation of the *Āryabhatiya*. The author has used Suryadeva's commentary and has also quoted verses from various other books. He has also referred to the *Mahabhaskariya-Bhasya-Vyakhya Siddhantadipika*, which proves that his commentary on the *Āryabhatiya* is a later work.

6. Nilakantha Somayaji's commentary

This commentary is known as, *Maha-Bhasya* and has been published in the *Trivandrum Sanskrit Series*, Nos. 101, 112 and 185. In this commentary, the author has

commented only upon the *Ganita, Kalakriya* and *Gola padas* of the *Āryabhatiya*.

(B) Commentaries in Telugu

7. Kodandarama's commentary

Kodandarama's commentary in Telugu is known as *Sudhataranga* and is on the first three *padas*, viz, *Gitika, Ganita* and *Kalakriya*. This commentary was edited by V. Lakshminarayana Sastri and published in the *Madras Government Oriental Series* in 1956.

8. Virupaksa's commentary

A manuscript of this commentary exists in the Oriental Manuscripts Library, Mysore.

(C) Commentaries in Malayalam

9. Krsnadasa's commentary

A manuscript of Krsnadasa's commentary covering the *Gitikapada* is available in the collection of K.V Sarma.

10. Commentaries by Ghatigopa

Ghatigopa wrote two commentaries in Malayalam, both of which are on the *Ganita, Kalakriya* and *Gola padas*. One commentary extends to 1850 *Granthas* (1*grantha* = 32 letters) and the smaller one to 1200 *granthas*.

Besides these commentaries on the *Aryabhatiya*, many papers on the *Āryabhatiya* and Āryabhata have been written in different national and international Journals. The *Āryabhatiya* had been translated by P.C. Sengupta in English in the year 1927, which was published in the Journal of the departments of letters, the University of Calcutta, volume XVI. PT.B. Mishra has written a commentary on the *Āryabhatiya*

in Hindi, which has been published by the Bihar Research Society, Patna in 1906.

3.5 The Āryabhata-Siddhanta

It is established beyond doubt, from the writings of Varahamihira (587 AD), Brahmagupta (628 AD), Bhaskara I (629 AD) and others, that *Āryabhata* wrote at least one book in addition to the *Āryabhatiya*. This book was on astronomy and known as *Āryabhata-Siddhanta*. The astronomical parameters and methods given in the *Āryabhata-Siddhanta* differed in some cases from those of the *Āryabhatiya*. Bhaskara I, in his *Mahabhaskariya*, has mentioned the important differences between the *Āryabhatiya* and the *Āryabhata-Siddhanta*. The astronomical parameters and methods ascribed to the *Āryabhata-Siddhanta* are generally the same as those found in Varahamihira's version of the *Surya-Siddhanta* and the *Sumati-Mahatantra* of *Acarya* Sumati of Nepal. Further, both these works are based on the *Surya- Siddhanta*. This proves that the *Āryabhata-Siddhanta* is an independent work like the *Āryabhatiya* and has the same relation with the *Surya-Siddhanta* as the *Āryabhatiya* has with the *Svayambhuva-Siddhanta*. Latadeva summarized the Surya-Siddhanta and revised it in the light of the *Āryabhata-Siddhanta*.

The *Āryabhata-Siddhanta* was a popular work and was used, not only as a text book of astronomy in North India, but also in every day calculations, such as those pertaining to marriage, nativity, etc. It was mentioned in the sixth century by Brahmagupta of Bhinma (Rajasthan) and Bhaskara I of Valabhi (Kathiawar), in the ninth century by Govinda-swami of Kerala, in the twelfth century by Mallikarjuna Suri of

Andhra and in the sixteenth century of Bhudhara of Kampilya and Tamma-Yajva of Andhra.

The great Brahmagupta was a bitter critic of *Āryabhat* at his early age, but was impressed by the popularity of the *Āryabhata-Siddhanta* and brought out an abridged edition of this work and called it *Khanda-Khandyaka*. Brahmagupta was not in complete agreement with the teachings of *Āryabhata*. So he planned his *Khanda-Khandyaka* in two parts. In Part I he summarized only the teachings of the *Āryabhata-Siddhanta*, while in Part II he gave accurate results with correctives and modifications to Part I. The *Khanda-Khandyaka* received wide acclaim and a large number of commentaries were written on it.

3.6 The Āryabhatiya and Āryabhata-Siddhanta

Bhaskara I gave the following tables which exhibit the main differences between the astronomical parameters of the *Āryabhatiya* and the *Āryabhata-Siddhanta*.

Table 2
Diameters and distances of planets in *yojanas*

	Āryabhatiya	*Āryabhata-Siddhanta*
Earth's diameter	1050	1600
Sun's diameter	4410	6480
Moon's diameter	315	480
Sun's distance	459585	689358
Moon's distance	34377	51566
Circumference of the sky	216000	324000
revolutions of the moon		

The number in the second and third columns are in the ratio 2:3, approximately. This is due to the fact that

the measures of *yojana* employed in the two works are in the ratio 3:2.

Table 3

Civil days, Omitted lunar days, and Revolutions of *Sighrocca* of Mercury and Jupiter in a period of 43,20,000 years

	Ā.ryabhatiya	*Aryabhata -Siddhanta*
Civil days	1,57,79,17,500	1,57,79,17,800
Omitted lunar days	2,50,82,580	2,50,82,280
Revolutions of *Sighrocca* of Mercury	1,79,37,020	1,79,37,000
Revolutions of Jupiter	3,64,224	3,64,220

Table 4

Longitudes of the planets' apogees (or aphelia) in 499 AD

Planet	*Āryabhatiya*	*Āryabhata-Siddhanta*
Sun	78°	80°
Mars	118°	110°
Mercury	210°	220°
Jupiter	180°	160°
Venus	90°	80°
Saturn	236°	240°

Table 5

Dimensions of planets' *manda* epicycles

	Āryabhatiya		*Āryabhata-Siddhanta*
	odd-quadrant	even-quadrant	
Sun	13°30′	13°30′	14°
Moon	31°30′	31°40′	31°
Mars	63°	81°	70°
Mercury	31°30′	22°30′	28°
Jupiter	31°30′	36°	32°

Venus	18°	9°	14°
Saturn	40°30′	58°30′	60°

The circumference of a planet's concentric or deferent (or mean orbit) is supposed to be 360 units (called degrees) in length and the above dimensions are on the same scale.

Table 6
Dimensions of planets' *Sighra* epicycles

	Āryabhatiya		*Āryabhata-Siddhanta*
	odd-quadrant	even-quadrant	
Mars	238°30′	229°30′	234°
Mercury	139°30′	130°30′	132°
Jupiter	72°00′	67°30′	72°
Venus	265°30′	256°30′	260°
Saturn	40°30′	30°00′	40°

Ramakrishna Ārādhya (1472 AD) has mentioned the astronomical instruments in thirty four verses from the chapter on the *Āryabhata-Siddhanta*. Some of these instruments were indeed devised by *Āryabhata*. The gnomon and the water instrument have generally been attributed to him. The commentators of the *Surya-Siddhanta* have praised Āryabhata because of these instruments devised by him.

Mallikarjuna Suri (1178 AD), Ramakrishna Aradhya (1472 AD) and Tamma Jajva (1599 AD) have referred to some special methods of the *Āryabhata-Siddhanta* and one of these methods is related to the determination of time approximately, from the shadow of the gnomon. It is a unique method and is not found in other works on Indian astronomy. It is described briefly as follows:

"When the sun is Scorpio, Capricorn or Aquarius, and it is within two *ghatir* from the moon, set up a gnomon of nine digits on the east-west line in such a way that the tip of its shadow may fall on the north-south line. Then the digits of the distance of the gnomon from the intersection of the east-west and north-south lines would approximately give the *ghatir* to elapse before noon or elapsed since noon (according as the observation is made before noon or after noon)."

3.7 The Surya-Siddhanta-Prakash

We come across several references to the third work of the Āryabhata, this is known as the *Surya-Siddhanta- Prakasa*, a commentary on the Surya-Siddhanta and is mentioned in the Descriptive Catalogue of Oriental Manuscript in the Mackenzie collection, London by H.H. Wilson, Madras 1882. However, nothing definite can be said. Since the manuscript of this work has not been found as yet. Perhaps on the basis of this some scholars are of the view that the Surya-Siddhanta was written by Āryabhata, but this view is far from the truth. Many more informations about the life of Āryabhata and his contributions to mathematics and astronomy will come to light, when the Surya-Siddhanta-Prakasa is unearthed.

CHAPTER 4

CONTRIBUTIONS OF ĀRYABHATA IN MATHEMATICS

4.1 Introduction

We have said in the beginning that Āryabhata made significant contributions to the development of mathematics in ancient India. *Ganita-pada* in the *Āryabhatiya* contains thirty-three *slókas* and all these *slókas* deal with mathematics. These *slókas* are so precise that in just thirty three *slókas* Āryabhata has described his entire findings in mathematics in brief. In these *slókas* he has dealt with square root, cube root, areas of different geometric shapes, right-angled triangles, quadratic equations, indeterminate equations of first order and other topics of mathematics.

However, we should not ignore the remarks made by Bhaskara I who has clearly said that whatever mathematics is included in *Āryabhatiya* is not a true representation of mathematics. The arguments given by Bhaskara I are based on certain historical facts and as such, these assertions seem quite valuable. In his own work he says:

> "In the *Ganita-pada* the *Acarya* (i.e Acarya Āryabhata) has dealt with the subject of mathematics by indications only, whereas in the

Kalakriya-pada and *Gola-pada*, he has dealt with 'reckoning of time' and 'spherical astronomy' in detail. So, the word *Ganita* (used by *Acarya* Āryabhata) one must understand 'a bit of mathematics.' Otherwise, the subject is very vast.

Āryabhata, perhaps, dealt with almost all the topics of mathematics known to exist in his time. According to Prof. Voladarsky [31], out of Āryabhata's contributions in the different branches of mathematics, "The most important are the rules for performing arithmetic operations according to the decimal place-value numeration. The earliest arithmetic rules known to us in this system were described by Āryabhata in the *Āryabhatiya*."

We shall, now, briefly describe Āryabhata's contributions to the different branches of mathematics.

4.2 Arithmetic

In his *Āryabhatiya*, Āryabhata has not talked about the fundamental operations of addition, subtraction, multiplication and division. Perhaps, the reason for this is that these operations were known to him and further, there was neither any scope for nor necessity of these operations in a precise book like *Āryabhatiya*.

Āryabhata has certainly talked about the square and cube of a number and also about the square root and cuberoot of a number. The square of a number is defined as the product of the number by itself. The cube of a number is defined as the product of the number by itself twice or as the product of the number by its square. The third *slóka* of *Ganita-pada* defines square and cube of a number as follows:

"वर्ग समचतुरश्रः फलं च सदृशद्वयस्य संवर्गः।
सदृशत्रय संवर्गो घनस्तथा द्वादशाश्रिः स्यात् ॥"

The method of finding the square root has been described in the fourth *slóka* of *Ganita-pada*. The *slóka* reads as follows:

भागं हरेदवर्गान्नित्यं द्विगुणेन वर्गमूलेन।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम् ॥

The fifth *slóka* of *Ganita-pada* describes the method of finding cube root of number as:

"अघनाद भजेद द्वितीयात त्रिगुणेन घनस्य मूलवर्गेण।
वर्ग स्त्रिपूर्व गुणितः शोध्यः प्रथमाद धनस्य घनात् ॥"

On going through these two *slókas* we can make the following observations:

(i) The method of finding the square root was not really first given by Āryabhata as it was found in earlier works like Sulva-works and Jaina mathematics. There are examples of finding the square root of big numbers in Jaina *Granthas*. However, the method of finding the cube root of a number was originally given by Āryabhata and it is not found in any earlier works.

(ii) These methods of finding the square and cube roots were used by Sridhana (900 AD) and other indian mathematicians.

(iii) These methods, as suggested by Āryabhata, are different from our present methods. The above mentioned fourth *slóka* of *Ganita-pada* describes the method of finding square root and consists of the following steps:

1. Subtract the greatest possible square from the last odd place.
2. Always divide the even place by twice, the square root of the preceding odd place.

3. Subtract from the odd place (standing on the right) the square of the quotient.
4. Repeat the process as long as there are still digits on the right.

This method is illustrated with the help of some examples.

Example 1: Find the squareroot of $X = 119716$

Solution: We have $9 < 11 < 16$ or $3 < \sqrt{11} < 4$, so that the integral part of $\sqrt{11}$ is 3. Marking the even places by horizontal lines and the odd places by vertical lines, we have

	1	–	1	–	1	⌊3
Subtracting the	11	9	7	1	6	
square of the root	-3^2					

Divide by $2 \times 3 = 6$	29	4
	24	
Subtracting the square	57	
of the quotient	-4^2	

Divide by $2 \times 34 = 68$	411	6
	408	
Subtracting the square	36	
of the quotient	-6^2	
	x	

As no more places are left on the right, the process ends.

∴ The required squareroot = 346.

Example 2: Find the squareroot of 738915489

Solution: Following the rule given by Āryabhata, we have

		1	–	1	–	1	–	1	–	1
Subtracting the square of the root	7	3	8	9	1	5	4	8	9	2
	-2^2									

Since integral of $\sqrt{7}$ is 2

Divide by $2 \times 2 = 4$ | 33 | 7

28

Subtracting the square of the quotient: 58 -7^2

Divide by $2 \times 27 = 54$ | 99 | 1

54

Subtracting the square of the quotient: 451 -1^2

Divide by $2 \times 271 = 542$ | 4505 | 8

4336

Subtracting the square of the quotient: 1694 -8^2

Divide by $2 \times 2718 = 5436$ | 16308 | 3

16308

Subtracting the square of the quotient: 09 -3^2

x

∴ The required square root = 7183.

Note: When divided by 4 we avoid to take 8 as at the next stage we will get $18 - 8^2 = -46$, a negative number.

Shri Bharathi Krishna Thirathji has a varlant which has much elegance and symmetry. It is based on the formula for the square of an integer in duptex notation.

Āryabhata gave an elegant for finding the cuberoot of a number. Let us call the unit place as *ghanna,* the 10-place "as the first *Aghana*" and the 10^2 place as "the Second *Aghana*" as illustrated in the following table:

Table 7

Second *Aghana*	First *Aghana*	Unit
10^2	10	1
10^4	10^3	
.	.	
.	.	
.	.	

Steps to find the cube root of a number:

a) Subtract the greatest possible cube from the last *ghana* place.
b) Divide the second *aghana* place by thrice the square of the cuberoot already obtained in (a).
c) Subtract from the first *aghana* place the square of the quotient multiplied by thrice of the previous cube root.
d) Subtract the cube of the quotient from the *ghana* place.
e) Repeat the process until all the digits are exhausted.

Example 1: Find the cube root of 11543176

Solution: Indicating the cubic place by vertical lines and the 1st and the 2nd non-cubic places by horizontal lines, we have

		I	–	–	I	–	–	I	
Subtracting the	11	5	4	3	1	7	6	
cube of the root	-2^3							2

Divide by $3 \times 2^2 = 12$ | 35 | 2

Subtracting the square of the quotient multiplied by thrice the previous root: 24

114

$-2^2 \times 3 \times 2$

903

Subtracting the cube of the quotient: -2^3 2

Divide by $3 \times (22)^2 = 1452$ | 8951 | 6

8712

2397

$-3.6^2.22$

216

-6^3 6

x

Since no more places are left in right, therefore the process ends and the required cube-root = 226.

Example 2: Find the cube root of 76928302277.

Solution: Indicating the cubic place by vertical lines and the 1st and 2nd non-cubic places by horizontal lines, we have

	1	–	–	1	–	–	1	–	–	1
Subtracting	76	9	2	8	3	0	2	2	2	7
the cube of	-4^3									
the root										

$4^3 < 76 < 5^3$

Divide by $3 \times 4^2 = 48$ | 129 | 2 | 4

96

332

$-2^2 \times 3 \times 4$

2848

-2^3 2

Divide by $3 \times (42)^2 = 5292$ | 28403 | 5

26460

19430

$-3 \times 5^2 \times 4^2$

162802

-5^3 5

Divide by $3 \times (425)^2 = 541875$ | 1626772 | 3

1625625

11477

$-3 \times 3^2 \times 425$

27

-3^3 3

x

The remainder being zero, the cube-root is exact.

Hence the required answer = 4 2 5 3

Now we give proofs of two theorems due to Āryabhata for finding the cube-root.

Theorem 4.2.1. Let a positive integer $X = x_1 10^3 + x_2 10^2 + x_3 10 + x_4$, where $0 < x_1 < 10^3$, $0 \le x_2, x_3, x_4 \le 9$ integers, then there exists an integer

a_2, $0 \le a_2 \le 9$ such that $X = (10a_1+a_2)^3 + X'$, where a_1 = $[\sqrt{x_1}]$; an integer $\le \sqrt{x_1}$.

Proof: We have $a_1 \le \sqrt[3]{x_1} < a_1 + 1$ so that $a_1^3 \le x_1 < (a_1 + 1)^3$. We remove the largest possible cube from x_1. Since $0 < x_1 < 10^3$, we must have $a_1 \le \sqrt[3]{x_1} < 10$ i.e. $1 \le a_1 \le 9$. It implies

$$x_1 = a_1^3 + b_1, \text{ where } b_1 = x_1 - a_1^3 \tag{1}$$

Dividing $10b_1 + x_2$ by $3a_1^2$ to get the quotient a_2 and remainder b_2, we have

$$10b_1 + x_2 = 3a_1^2 a_2 + b_2 \tag{2}$$

Subtract $3a_1 a_2^2$ from $10b_2 + x_3$ to get b_3, thus

$$10b_2 + x_3 = 3a_1 a_2^2 + b_3 \tag{3}$$

Subtract a_2^3 from $10b_3 + x_4$ to get b_4, thus

$$10b_3 + x_4 = a_2^3 + b_4 \tag{4}$$

From (1), (2), (3) & (4), we get

$$\left.\begin{aligned} x_1 &= a_1^3 + b_1 \\ 10b_1 + x_2 &= 3a_1^2 \cdot a_2 + b_2 \\ 10b_2 + x_3 &= 3a_1 \cdot a_2^2 + b_3 \\ 10b_3 + x_4 &= a_2^3 + b_4 \end{aligned}\right] \tag{5}$$

Multiplying the equations (5), in order, by 10^3, 10^2, 10 and 1 and adding, we have

$$x_1 10^3 + x_2 10^2 + x_3 10 + x_4 = a_1 10^3 + 3(a_1 \cdot 10)^2 a_2 + 3(a_1 \cdot 10) a_2^2 + a_2^3 + b_4$$

$$= (a_1 10 + a_2)^3 + b_4$$

Putting $b_4 = X'$, we get the required result. Hence the theorem is proved.

Theorem 4.2.2 Given a positive integer

$$X = x_1\ 10^9 + x_2\ 10^8 + \ldots + x_9\ 10 + x_{10} \qquad (6)$$

$$1 \le x_1 \le 10^3, 0 \le x_k \le 9 \quad (k = 2, 3, \ldots, 10)$$

Solve the following systems of equations for integers a_1, a_2, a_3, a_4 such that $1 \le a_1 \le 9, 0 \le a_2, a_3, a_4 \le 9$:
(A)

$$\left.\begin{aligned} x_1 &= r_1^3 + b_1 \\ 10b_1 + x_2 &= 3r_1^2 a_2 + b_2 \\ 10b_2 + x_3 &= 3r_1 a_2^2 + b_3 \\ 10b_3 + x_4 &= a_2^3 + b_4 \end{aligned}\right] \quad \begin{aligned} r_1 &= a_1 \\ X_1 &= x_1\ 10^3 + x_2\ 10^2 + x_3\ 10 + x_4 \end{aligned}$$

(B)

$$\left.\begin{aligned} X_1 &= r_2^3 + b_4 \\ 10b_4 + x_5 &= 3r_2^2 a_3 + b_5 \\ 10b_5 + x_6 &= 3r_2^2 a_3^2 + b_6 \\ 10b_6 + x_7 &= a_3^3 + b_7 \end{aligned}\right] \quad \begin{aligned} r_2 &= 10r_1 + a_2 \\ X_2 &= x_5\ 10^2 + x_6\ 10 + x_7 \end{aligned}$$

(C)

$$\left.\begin{aligned} X_1 10^3 + X_2 &= r_3^3 + b_7 \\ 10b_7 + x_8 &= 3r_3^2 a_4 + b_8 \\ 10b_8 + x_9 &= 3r_3 a_4^2 + b_9 \\ 10b_9 + x_{10} &= a_4^3 + b_{10} \end{aligned}\right] \quad \begin{aligned} r_3 &= 10r_2 + a_3 \\ X_3 &= x_8\ 10^2 + x_9\ 10 + x_{10} \end{aligned}$$

Thus we claim

$$X = (a_1\ 10^3 + a_2\ 10^2 + a_3\ 10 + a_4)^3 + b_{10} \qquad (7)$$

It implies that

X is a perfect cube if and only if $b_{10} = 0$

Proof: Applying theorem 4.2.1, to the equations (A), we have

$$X_1 = (10r_1 + a_2)^3 + b_4 = r_2^3 + b_4$$

Then, by equations (B), we get

$$X_1\ 10^3 + x_5\ 10^2 + x_6\ 10 + x_7 = (10r_2 + a_3)^3 + b_7 = r_3^3 + b_7$$

or

$$X_1\ 10^3 + x_2 = r_3^3 + b_7$$

Finally by equations (C), we have

$$(X_1\ 10^3 + X_2)\ 10^3 + x_8\ 10^2 + x_9\ 10$$

$$= (10\ r_3 + a_4)^3 + b_{10}$$

$$= (10^2\ r_2 + 10a_3 + a_4)^3 + b_{10}$$

$$= (a_1\ 10^3 + a_2\ 10^2 + a_3\ 10 + a_4)^3 + b_{10}$$

which is (7). Hence, the theorem is proved.

Example: Find the cube root of x = 792994249216.

Solution: Applying Theorem 4.2.2, we have

$$X_1 = 792994249$$

$$X_1\ 10^3 + X_2 = 792994249$$

$$X_1\ 10^6 + X_2\ 10^3 + X_3 = 792994249216$$

Table 8

k	1	2	3	4	5	6	7	8	9	10
x_k	792	9	9	4	2	4	9	2	1	6
b_k	63	153	1431	14306	16102	154124	1541124	9992	21	0
a_k	9	2	5	6						
r_k	9	92	925	9256						

The method of computation of three graphs (A), (B) and (C) in Table 8 is as follows:

3×9^2	792	9
	-9^3	
	639	2
	$-3 \times 9^2 \times 2$	
	1539	
	$-3 \times 9 \times 2^2$	
	14314	
	-2^3	
$b_4 =$	14306	

$3 \times (92)^2$	143062	5
	$-3 \times (92)^2 \times 5$	
	161024	
	$-3 \times 92 \times 5^2$	
	1541249	
	-5^3	
$b_7 =$	1541124	

$3 \times (925)^2$	15411242	6
	$-3 \times (925)^2 \times 6$	
	99921	
	$-3 \times 925 \times 6^2$	
	216	
	-6^3	
$b_{10} =$	0	

If we compare Āryabhata's methods of finding square root and cube root of a number with those followed in modern times, we can say that the

Āryabhata's methods are similar to modern methods with the only difference in the number of steps involved in the process. In fact the present method of finding the cube- root is a contraction of Āryabhata's method. The extracting of cube-root was not known in India before Āryabhata. All the mathematicians after him followed his method with some modifications here and there.

We may conclude from the methods of extracting square-root and cube-root as described by Āryabhata and discussed by us that Āryabhata was perfect in dealing with the decimal place-value notation. F. Cajori refers to Āryabhata's knowledge of the place-value notation. He observes, "In this connection Āryabhata speaks of dividing a number into period of two and three digits. From this we infer that the principle of position and the zero in the numerical notation were already known to him." Similar view has been expressed by the editors of *Encyclopaedia International.* Hence Kaye's assertion that "There is not in any part of Āryabhata's work the remotest indication of a notation with 'place value'." This only shows Kaye's extreme prejudice and a lack of proper understanding of ancient Indian mathematics.

In the twenty-fifth slóka of *Ganita-pada,* Āryabhata has described the method of finding the interest on any given amount. It says:

"मूलफलं सफलं कालमूल गुणामर्धमूल कृतियुक्तम्।
तन्मुलं मूलार्धोन कालहृतं स्वमूलंफलम् ॥"

Thus, in finding the interest on an amount *x* in time *t* we take the following steps.

1. Add the interest on *x* and the interest on the interest. Call it *A.*
2. Multiply this sum *A* by *x,* call it *B.*

3. To this multiplication *B*, add $x^2/4$ (the square of half of the given amount), call it *C*.
4. Take the square root of this final quantity *C*. Call it *D*.
5. Subtract *x* from *D*/2. Call it *E*.
6. Divide *E* by *t*.

Whatever is now obtained is the required interest. We illustrate this method by the following example:

Example: If the interest on Rs 100 for one month alongwith its interest for 6 months is known to be 16, find the rate of interest on the principal.

Solution: Applying the above rule, we have

$$\text{rate} = \frac{\sqrt{16 \times 100 \times 6 + 50^2} - 50}{6}$$

$$= \frac{\sqrt{9600 + 2500} - 50}{6} = \frac{\sqrt{12100} - 50}{6} = \text{Rs } 10.$$

This is one month interest on the principal i.e. the rate of interest.

The twenty sixth *slóka* of *Ganita-pada* has described rule three. It reads as:

"त्रैराशिक फलराशिं तमथेच्छाराशिना हतं कृत्वा।

लब्धं प्रमाणभाजितं तस्मादिच्छाफलमिदं स्यात् ॥"

The slóka means "In the rule of three, the fruit being multiplied by the requisition is divided by the arguments. The quotient is the fruit corresponding to the requisition."

The rule is simple and clear. There are three terms in it, namely, argument, fruit and requisition. When these three are given, the fourth term, the fruit corresponding to the requisition can be determined as follows:

$$\text{Fruit corresponding to requisition} = \frac{\text{fruit} \times \text{requisition}}{\text{argument}}$$

Both Indian and western scholars have appreciated this rule because of its simplicity and applicability to the problems of arithmetic. Bhaskara II calls this rule 'The arithmetic itself'. Some European writers call it the 'Golden Rule'. Hodder of the seventeenth century writes in praise of this rule that "for as gold tremeends all other metals, so doth this rule all others in Arithmetick."

It is the *Āryabhatiya* where we first find the statement by the rule of three. Āryabhata could not give examples, however, the later Hindu mathematicians, used the rule in solving a number of problems in arithmetic. Smith's statement is worth mentioning. He remarks that "the mercantile rule of three seems to have originated among the Hindus." In this regard Datta and Singh [11] observe that "it was transmitted to the Arabs probably in the eighth century and hence travelled to Europe, where it was held in very high esteem and called the 'Golden Rule'". In the west, this rule was used by Widman (1489 AD) in his works and was followed by Tonstall (1522 AD), Frisius (1540 AD), Trenchant (1566 AD) and others. Thus, Āryabhata paved the path for further development in arithmetic by giving the rule of three.

Āryabhata also gave the method of inversion in which the process of operation starts from the end and the working is backward while in other rules, the operation starts right from the beginning and goes forward. The technique of the method of inversion is that the fundamental operations mentioned in the problem are reversed i.e, if 'addition' is mentioned, it will mean 'subtraction' and vice-versa; similarly if the multiplication is named, it will mean division and vice-versa, and so on. This was a "favourite method" in

ancient India and is still in practice, while in western countries it became popular in the sixteenth century. The 'method of inversion' has been described in the twenty eighth slóka of the *Āryabhatiya*. Hence, this definitely has origin in India.

The decimal system was known to Āryabhata and it was used by him. In the words of Professor Ram Behari [6] "He was the first Indian mathematician to adopt the decimal place-value system and to write formally on the subject of mathematics." Thus we conclude that Āryabhata gave a number of new results in arithmetic.

4.3 Algebra

"Āryabhata had described, in the form of highly condensed rules, his mathematical ideas in his treatise, the *Āryabhatiya*. He is known as the first individual (mathematician) who could be rightly called the father of algebra in India," says Prof. Ansary [1].

In the thirtieth *slóka* of *Ganita-pada*, Āryabhata describes algebra and therefore, some people regard Āryabhata as the founder of algebra: This *slóka* reads:

गुलिकान्तरेण विभजेद् द्वयोः पुरूषयोस्तु रूपकविशेषम्।
लब्धं गुलिका मूल्यं यद्यर्थकृतं भवति तुल्यम् ॥

It says that if the difference in known quantities of two persons is divided by the difference in their unknown amount, the result shall give the value of the unknown quantity. In his commentary on the *Āryabhatiya*, Paramesvar (1430 AD) writes:

अव्यक्तमूल्यानां मूल्यप्रदर्शनर्मित्याह। गवादिदव्यं गुलिकाशब्दे
नोच्यते रूपकशब्देन पणादि-संक्षित स्वाणीदि द्रव्यम् ॥

Parmesvar further illustrates Āryabhata's method in the following example:

"समस्वया रूपकाणां शतंषाष्टि कमाद्धनम्।
गावष्ष्डवाणिजश्चाष्टौ तत्र गोमूल्यकं कियत् ॥"

This translates as "Let there be two businessmen. One has rupees hundred and 6 cows, while the other has rupees sixty and 8 cows. If they possess the same amount of property, what is the price of one cow?."

Here, according to Āryabhata, if x is the price of a cow in rupees, then

$$x = \frac{\text{difference in known quantity}}{\text{difference in unknown quantity}}$$

The known quantity is "rupees" and the unknown quantity is the price of one "Cow". Thus,

$$x = \frac{100 - 60}{8 - 6} = \frac{40}{2} = 20$$

Hence, the price of one cow must be rupees 20.

We have already quoted the twenty-fifth *slóka* of *Ganita-pada*. This *slóka* is of great historical importance, not only because it gives the method of finding interest on a given amount, but also because of the fact that in the process of finding interest, the use of quadratic equations has to be made. This further confirms our claim that Āryabhata was the orginator of algebra.

It was, perhaps, Āryabhata who gave the following identities for the first time.

(i) $a + (a + d) \ldots (a + (n - 1)\, d) = \frac{n}{2} [2a + (n - 1)\, d]$

This has been stated in the following nineteenth *slóka* of the *ganita-pada* of the *Āryabhatiya*:

"इष्टं व्येकं दलितं सपूर्व मुत्तर गुणं समुखमध्यम्।
इष्टग्रणितमिष्ट धनं त्वथवाद्यन्तं पदार्धहतम् ॥"

(ii) $$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) $$1^3 + 2^3 + \ldots + n^3 = \left[\frac{1}{2} n(n+1)\right]^2$$

(iv) $$1 + (1+2) + (1+2+3) + \ldots + (1+2+\ldots+n)$$

$$= n(n+1)(n+2)$$

(v) If S is the sum of the series,
$S = a + (a+d) + (a+2d) + \ldots$ upto n terms, then

$$n = \frac{1}{2}\left[\frac{\sqrt{8ds + (2a-d)^2} - 2a + d}{d}\right]$$

This has been described in the following twentieth *slóka* of the *Ganita-pada* of the *Āryabhatiya*:

गच्छोऽष्टोत्तर गुणिताद् द्विगुणाद्युत्तर विशेष वर्गयुतात्।
मूलं द्विगुणाद्यूनं स्वोत्तर भजितं सरूपाधर्म् ॥

The contributions of Āryabhata in the field of equations and their solutions are enormous. He has obtained the solutions of the following type of equations:

1. Let $x - y = m$ and $xy = n$, then find x and y.

The solution of this equation is given in the twenty-fourth *slóka* of the *Ganita-pada* of the *Aryabhatiya*. It is as follows:

द्विकृति गुणात् संवर्गाद द्वयन्तवर्गेण संयुक्तान्मूलम।
अन्तर युक्तं हीनं तद् गुणाकार द्वयं दलितम् ॥"

It translates as, "Multiply the product by four, then add the square of the difference of the two quantities,

and then take the square root. Set down this square root in two places. In one place increase it by the difference of the two quantities and in the other place decrease it by the same. The results thus obtained when divided by two, give the two factors of the same product." Thus, the solution will be

$$x=\frac{\sqrt{4n+m^2}+m}{2} \quad \text{and} \quad y=\frac{\sqrt{4n+m^2}-m}{2}$$

2. $ax \pm b = cx \pm d$
3. $ax - by = c$

The thirty-second and thirty-third *slókas* in the *Ganita-pada* are probably the first to explain the method which is known as the method of *kuttkart*. The *slókas* are:

"अधिकाग्र भागहारं छिन्द्यादूनाग्र भागहारेण।

शेष परस्पर भक्तं मतिगुणमग्रोरन्तरे क्षिप्तम् ॥

अध उपरिगुणित मन्तय युगूनाग्रच्छेदभाजितेशेषम्।

अधिकाग्रच्छेद गुणं द्विच्छेदाग्रमधिकाग्रयुतम ॥"

The result stated in these *slókas* is ment for solving a residual pulveriser (*sagra-kuttakava*), i.e. a problem of the type given below:

A number leaves 1 as the remainder when divided by 5, and 2 when divided by 7. Calculate what that number is.

For a detailed discussion, of this problem refer to Satya Prakash [21]. Aryabhata is best known in Algebra for solving this indeterminate equation. This equation is now known as the Diphantine Equation and historians give credit to Diphantus for solving this equation. This is a historical error and it seems that these historians were not familiar with the solutions provided by Āryabhata, much before Diphantus.

4. $ax^2 + bx + c = 0$

5. $x_2 + x_3 + x_4 = a_1$

$x_3 + x_4 + x_1 = a_2$

$x_4 + x_1 + x_2 = a_3$

$x_1 + x_2 + x_3 = a_4$

Apart from these equations, Āryabhata was the first, who introduced the rules to solve some other types of indeterminate problems. Prof. Ram Behari [6] has said, "... Āryabhata was the world's first mathematician to solve the indeterminate equation of the first degree."

4.4 Value of π

The value of π, as given by Āryabhata, is really a wonderful contribution. It is important to know that the value (approximate) of π was known in India even about 100 BC. In the "*Manav Shulva Sutra*" it has been quoted as 3.16049 and this value is correct only upto the first decimal point. Boddhayan (800 BC) has given two values of π, namely, 3 and 3.0885. According to Shatpati Brahman, the value of π is 3.0805 or 3.0883 or 3.004. According to Archimedes (287–222 BC), the value of π is given by 22 / 7 = 3.1428, which is correct upto two decimal places. Brahmagupta (628 AD) thought that the value of π is $\sqrt{10}$. The best approximated value of π was given by Āryabhata to be 3.1416, approximately. The tenth *slóka* in the *Ganita-pada* of Āryabhata says:

चतुरधिकंशतमष्टगुणं द्वाषष्टिस्तथा सहस्त्राणाम्।

अयुतद्वय विष्कम्भस्यासन्नो वृत्त परिणाहः ॥

It translates as, "100 plus 4, multiplied by 8, and added to 62000: this is the approximate measure of the cir-

cumference of a circle whose diameter is 20,000." Thus, according to Āryabhata, if the diameter of a circle is 2000 and the circumference is 62832, then the value of π is given by 62832 / 2000 = 3.1416. This value is correct upto four decimal places and is being used in computation nowadays. This value is certainly better than the value given by the Greek and other Indian mathematicians. Some persons believe that possibly Āryabhata had the idea of irrational numbers. Some commentators believe that the value 3.1416 of π was first given by Archimedes, and it is not correct because Archimedes placed π in between $3\frac{1}{7}$ and $3\frac{10}{11}$. Prof. Kurt Elfering [12] says, "Moreover, it is of great importance that Āryabhata taught the uncommonly exact value of π = 3.1416," S.K Jha and M. Jha [17] also made a study of the value of π known to ancient and Jain mathematicians.

In this paragraph it is discussed whether Āryabhata used the value of π, notwithstanding with its high degree of accuracy in his works or not. The authors are of the opinion that he applied it in his works several times, for example, for the area of a circle and for the volume of a sphere which require the value of π. He has also used it in the preparation of a scientific table for the differences of Sines and in calculating the dimensions of the earth. In addition to this there are some other examples found in the *Āryabhatiya*, which confirm the fact that Āryabhata has used the value of π at several places. However, there are some western scholars who believe that Aryabhata did not use the value of π. Possibly they might have overlooked its usage in the *Āryabhatiya* or could not interpret its applications correctly. G.R. Kaye was wrong in making the observation that "But it is rather extraordinary that Āryabhata himself never utilized this value."

One important thing about Āryabhata's value of π is his idea of π as an irrational number. Nilakantha, one

of the commentators of the *Āryabhatiya* conclude that Āryabhata did not only give the value of π = 3.1416, but also realized that π was an irrational number. It is believed that the ancient Hindus know about the existence of irrational numbers and use of arithmetic operations on them. Simple surd numbers have been frequently used in *Sulva-Sutra* and this might have induced Āryabhata to consider π as an irrational number. He gave the approximate value of π. Consequently, the ratio of the circumference to the diameter could never be expressed as the ratio of two integers. In this way he anticipated the discovery of π as an irrational number 1300 years ago. Its formal proof was given by Lambert in 1761 AD.

4.5 Mensuration and Geometry

It is said that the Greek mathematician Thales (600 BC) is the father of geometry. But there is evidence that Indians had the knowledge of geometry much before Thales (*Shulva Kal* 1200 BC – 800 BC.). However, it is true that the Greeks made tremendous progress in the field of geometry and India was left far behind. Perhaps, the reason for this was that the Greeks were more geometry-minded, while the Indians were more inclined towards arithmetic and algebra. That is why, Āryabhata has referred to squares and cubes in arithmetic.

Aryabhata has dealt with the square, the cube, the triangle, the trapezium, the circle and the sphere. The sixth slóka in the *Ganita-pada* of the *Āryabhatiya* defines the area of a triangle. It says,

त्रिभुजस्य फलशरीरं समदल कोटी भुजर्धसंवर्ग:।
ऊर्ध्व भुजा तत्संवर्गार्ध स धन: षडश्रिरिति ॥

This means, "the product of the perpendicular drawn from the vertex and half the base gives the measures of the area of triangle. Half the product of that area (of the triangular base) and the height is the volume of a six-edged solid." Āryabhata has probably given the area of an isosceles triangle, as the above *slóka* says that the area of a triangle is equal to half of the product of the base and perpendicular from the vertex which bisects the base. The area of a circle and the volume of a sphere is given in the following *slóka*.

"समपरिणाहस्यार्घ विष्कं भार्घहतमेव वृत्तफलम्।
तन्निजमूलेन हतं घन गोल फलं निखशेषम् ॥"

The area of a trapezium was defined as the product of half the sum of the parallel sides and the altitude. The value of a pyramid was given as the product of the area of its base and the altitude. Aryabhata has further stated two theorems. These are:

1. The so called Pythagoras theorem which was known to Indians since the time of the *Sulva-sutra*. It states that in a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the perpendicular and the base.

2. If the diameter AC and chord XY in a circle intersect at right angles, then

$$AB \times AC = \frac{(XY)^2}{2},$$

where B is the point of intersection of AC and XY. The rule given by Āryabhata for finding the volume of right pyramid is as follows:

ऊर्ध्वभुजातत्संवर्गार्धं स घनः षडश्रिरिति।

It means that half the product of that area and the

height is the volume of a six-edged solid. This rule was inaccurate and the correct one is found in the *Brahma-Sphuta-Siddhanta* of Brahmagupta, which states, "The volume of a uniform excavation divided by three in the volume of needle shaped solid, i.e., the volume of a cone or pyramid = 1/3 (area of base) × height." Bhaskara I seems to have been unaware of this formula, for he has no comment to make on the rule of Āryabhata.

The formula for finding the volume of a sphere was given in the seventh *slóka* of the *Ganita-pada*, and it states that if r be the radius of a sphere, then, according to Āryabhata,

$$\text{the volume of a sphere} = \pi r^2 \sqrt{\pi r^2}$$

The rule for finding the area of a trapezium was given in the eighth *slóka* whose English translation is given as, "Multiply the base and the face of the trapezium by the height, and divide each product by the sum of the base and the face. The results are the lengths of the perpendiculars on the base and the face from the point of intersection of the diagonals. The result obtained by multiplying half of the sum of the base and the face by the height is to be known as the area of the trapezium."

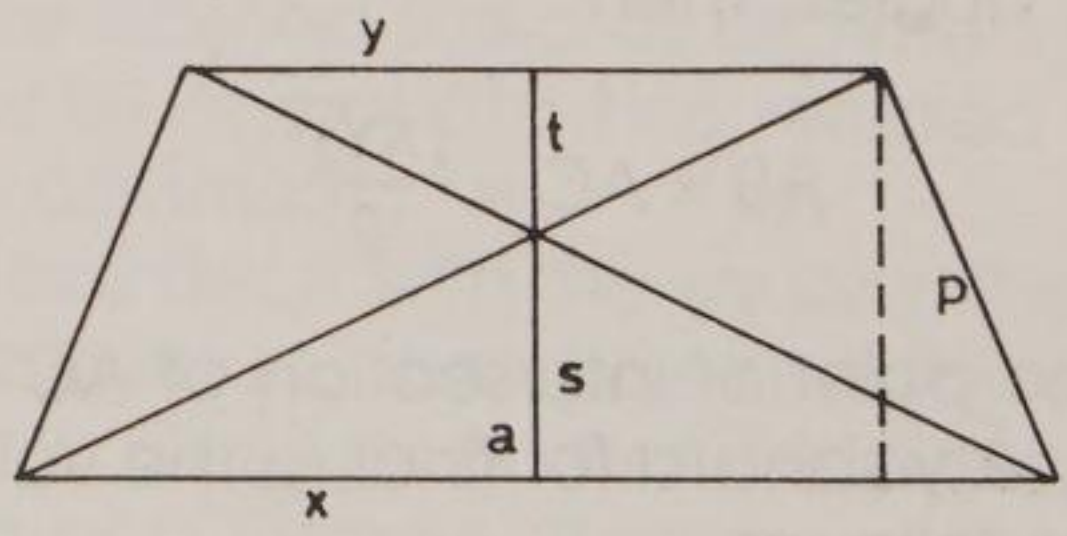


Fig. 3

Let x, y be the base and the face, p the height, and s, t the lengths of the perpendiculars from the

point of intersection of diagonals on the base and the face. Then

$$s = \frac{xp}{x+y}, \quad t = \frac{yp}{x+y}$$

and $$\text{area} = \frac{1}{2}(x+y)\,p$$

The volumes of a sphere and a pyramid were not correctly given by Āryabhata, while the Greeks had already given the correct values by the time of Āryabhata. This, in fact, proves that Āryabhata was not aware of the work of the Greeks in geometry and therefore, whatever he did, was original. Prof. Engering [12] says, "The two stanzas, in which Āryabhata dealt with the triangle, pyramid, circle and sphere, had special importance for ancient Indian geometry, because the volumes of the pyramid and the sphere seem to be incorrectly stated, and indeed, a gramatically correct translation leads to a false result for these volumes."

Therefore, although scholars have especially admired the arithmetical and algebraic achievements of the ancient Indians' yet, they have some doubt about their geometrical understanding. There is some controversy about the rule for finding the volume of a sphere given by Āryabhata. Prof. R.C. Gupta [15] has rightly noted that "However, by giving very unusual interpretations, some scholars maintain that the rule in the text is not about the volume of a sphere, rather it is the rule for the surface of a hemisphere, for which he gives a correct expression."

Even Āryabhata's inaccurate formula for the volume of a sphere is significant from the historical point of view. Moreover, it motivated his followers to find out the correct formula. Sridhara gave its correct value, which was adopted and explicitly expressed by

Bhaskara II. Bhaskara II's formula is correct and is expressed as follows:

$$\text{Volume of the sphere} = \frac{4}{3}\pi r^3$$

where r is the radius of the circle.

4.6 Trigonometry

Āryabhata has given a table of R sine-differences at equal intervals of 225′, where R = 3438′. While on the one hand it is still not very clear how Āryabhata prepared the table, on the other hand it is under dispute as to whether this table was indeed given by Āryabhata or taken from the *Surya Siddhanta*. However, one thing is clear and that is that this was indeed a great contribution made by the Hindus.

In Aryabhata's table of R sine-differences, the value sine (3 ¾)° and its multiples are given. The sines of 0° to 90° are determined by interpolation. Since sine (3 ¾)° = sine 225′ is very small, we consider chord and arc equal and put sine (3 ¾)° = 225′. The other R-sines are found out on this basis. In Āryabhata's table the values of another function namely, 'versed sine,' are also given. It is necessary to see what the versedness in inversed-sine is, because we take versed sine as (1 – Cos θ), while its meaning is 1/Sin θ and there is no explanation for this error of meaning. In fact, versed-sine is the translation of the Sanskrit word 'उत्क्रमज्या'

Table 9
R-Sine Table

S. No.	Angle	Hindu-sines	Hindu versed-sines r–sin(90° – θ)	Hindu sine-differences
1	3°45′	225	7	224
2	7°30′	449	29	220
3	11°15′	671	66	219
4	15°	890	117	215
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
23	80°15′	3431	3213	7
24	90°	3438	3438	

From the table given above, it can be said that the formula, sine $(90 - \theta)$ = Cos θ was already known through the medium of $r -$ Cos θ (r = 3438).

There are two methods given in the *Āryabhatiya* for computing the R-sines of arcs at equal intervals. One is the geometrical method and the other is based on the formula given in the following *slóka* of the *Āryabhatiya*:

प्रथमाच्चापज्यार्धादं यैरुनंखण्डितं द्वितीयार्धम्।
तत्प्रथम ज्यार्धाशैस्तै स्तैरूनानि शेषाणि ॥

The geometrical method for computing the R-sines of arcs at equal intervals is explained as follows:

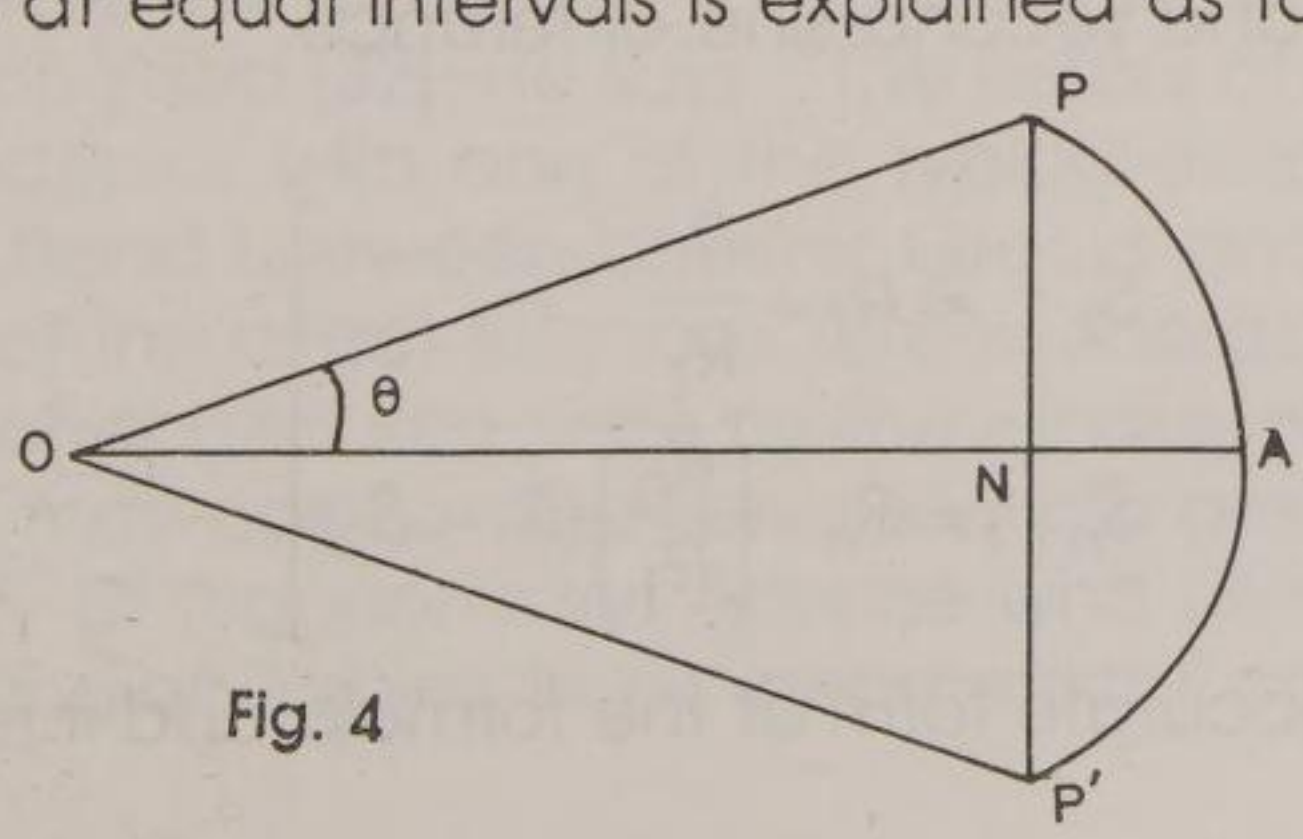


Fig. 4

Let PAP′ be any arc of a circle with centre O and A be the middle point of arc PP′ and N be the mid-point of chord PP′. Then, PN or NP′ is called Hindu sine of angle θ. If radius OP = r and ∠ AOP = θ,

$$\text{Hindu sine } \theta = \text{NP} = r \text{ sine } \theta$$

$$\text{Modern sine } \theta = \frac{\text{Hindu sine } \theta}{r} \qquad (1)$$

The value of r was found out and was equal to 3438′. Other Hindu sine θ for different angles θ can be determined by the method of interpolation.

$$S_p = S_{p-1} + S_1 - \sum_{i=1}^{p-1} q_i \qquad (2)$$

where S_1, S_2 ... S_p are values of Hindu sines and

$$q_i = \frac{S_i}{225}$$

Datta and Singh [11] have translated the above *slóka* in following way:

"The first R-sine divided by itself and thus diminished by the quotient will give the second sine difference. For computing any other sine difference, the sum of all the preceding sine differences diminished by the product of the quotient of the preceding R-sine by the first R-sine and the difference between first sine difference and second sine difference."

That is

$$\left.\begin{aligned} S_2 &= R_1 - \frac{R_1}{R_1} \\ S_{n+1} &= S_n - \left(\frac{R_n}{R_1}\right)(S_1 - S_2) \end{aligned}\right] \qquad (3)$$

This is the accurate form of the formula and it reduces

to form (2), since $S_1 - S_2 = 225' - 224' = 1'$. The following is the trigonometrical rationale of (3):

$$S_n - S_{n+1} = \{R \sin nh - R \sin (n-1)\, h\}$$

$$\therefore \{R \sin (n+1)\, h - R \sin nh\}$$

where $h = 225'$

$$= 2R \sin nh - \{R \sin (n+1)\, h + R \sin (n-1)\, h\}$$

$$= 2R \sin nh - \frac{2R \sin nh\, R \cos h}{R}$$

$$= 2R \sin nh \times \frac{(R - R \cos h)}{R}$$

$$= R \sin nh \frac{2R \sin h\, (R - R \cos h)}{R \sin h\, R}$$

$$= \left(\frac{R_n}{R_1}\right)(S_1 - S_2)\,,$$

where

$$S_1 - S_2 = \frac{2R \sin h \times (R - R \cos h)}{R}$$

$$R_n = R \sin nh \quad \text{and} \quad R_1 = R \sin h$$

Before the close of this section we give the remarks of Prof. Yana [32]. He says, "The history of trigonometry provides us with one of the typical instances of cultural bond between ancient Greeks and India in the field of the exact sciences. Greek chords were halved and changed into sines by the Hindus. However, simple it may appear, the device was a great step in the history of trigonometry. Anyone who uses both Greek and Indian tables in his geometrical calculation will

immediately realise the far greater convenience of the latter."

4.7 Construction of Circle and the Shadow Sphere

वृतं भ्रमेण साध्यं त्रिभुजं च चतुर्भुजं च कर्णाभ्याम्।
साध्या जलेन सममूरधऊर्ध्व लम्बकेनैव ॥

The above *slóka* translates as, "a circle should be constructed by means of a pair of compasses; a triangle and a quadrilateral by means of the two hypotenuses. The level of ground should be tested by means of water; and verticality by means of a plumb."

In a triangle the two hypotenuses are the two lateral sides above the base; in the case of a rectangle, these are the two diagonals; and in the case of a trapezium, these are the two lateral sides. In fact, it is referred to the usual methods of constructing a triangle, parallelogram and trapezium, when three sides in case of triangle; one side and two diagonals in case of parallelogram; the base, height and the two lateral sides in case of trapezium are given.

Āryabhata also discussed the methods of testing of level and verticality. As regards testing the level of the ground, Bhaskara I observes, "When there is no wind, place a jar (full) of water upon a tripod on the ground which has been made plane by means of eye or thread, and bore a hole at the bottom of the jar, so that water may have a continuous flow. Where the water falling on the ground spreads in a circle, there the ground is in perfect level, where the water accumulates after departing from the circle of water, it is low, and where the water does not reach, it is high."

शङ्कोः प्रमाणवर्ग छायावर्गेणं संयुतं कृत्वा।

यत्तस्य वर्गमूलं विष्कम्भार्घ स्ववृत्तस्य ॥

It means, "Add the square of the height of the gnomon to the square of its shadow. The square root of that sum is the semi-diameter of the circle of shadow."

According to Bhaskara I, the Hindu astronomers had different opinions about the shape of a gnomon. Some took a gnomon with one third at the bottom of the shape of a right prism on a square base, one third in the middle of the shape of a cylinder and one third at the top of the shape of a cone. Others took a gnomon of the shape of a right prism on a square base. Bhaskara I, further, writes that the followers of Āryabhata had preference for a cylindrical gnomon, made of excellent timber, free from holes, knots and sears with large diameter and height. In order to get the best tip of the shadow, a cylindrical needle made of timber or iron was fixed vertically in the middle of the top of the gnomon. Such a gnomon, being large and massive, was unaffected by the wind. It was cylindrical and surrounded by a needle of small diameter, so it was easy to manufacture and the tip of the shadow was also easy to perceive.

A gnomon was generally divided into twelve equal parts called angulas. However, there was no such hard and fast rule, according to Bhaskara I, a gnomon could be of any length with any number of divisions.

शङ्कुगुणं शङ्कुभुजविवरं शङ्कुभुजयोर्विशेषहृतम्।

यल्लब्धं या छाया ज्ञेया शङ्कोः स्वमूलाद्धि ॥

The above *slóka* translates as, "Multiply the distance between the gnomon and the lamp-post by the height of the gnomon, and divide the product by the difference between the height of the lamp-post and the gnomon. The quotient should be known as the

length of the shadow, measured from the foot of the gnomon."

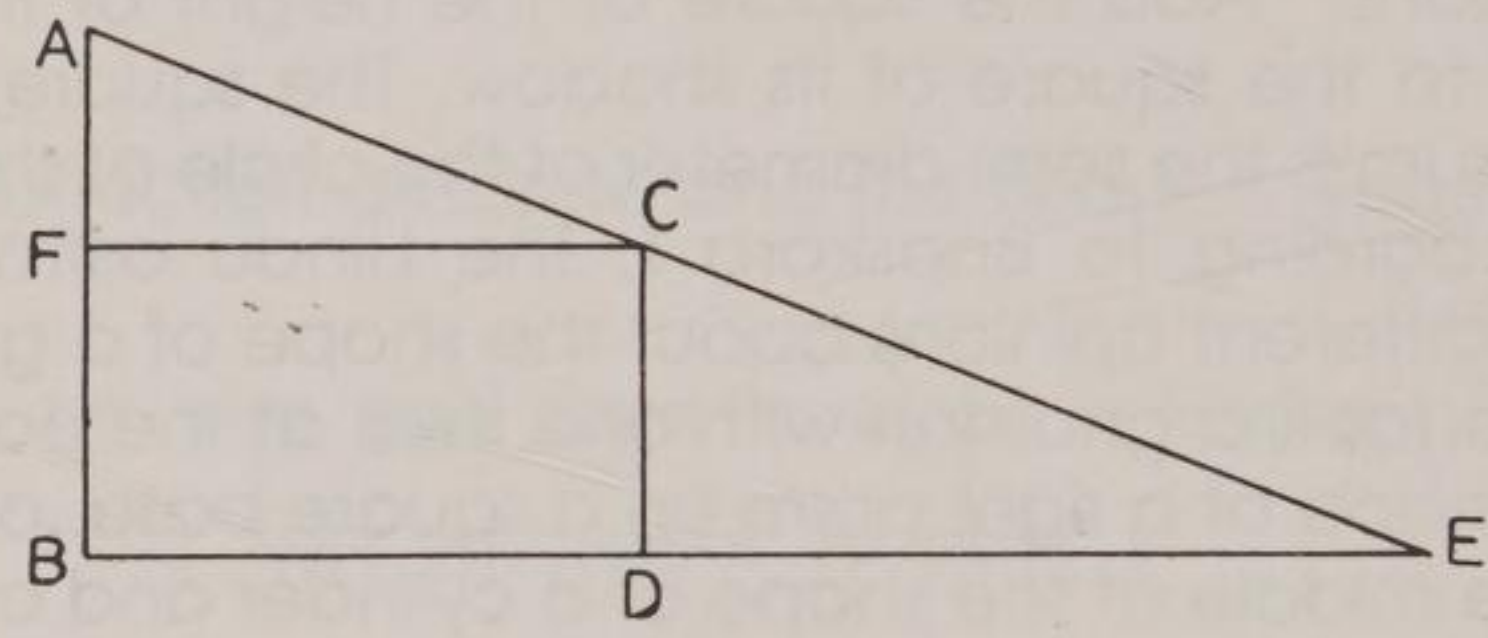


Fig. 5

Let AB be the lamp-post, CD the gnomon and E the point where AC and BD produced meet. Then DE is the shadow cast by the gnomon, due to light from the lamp at A.

Let FC be parallel to BD. Then, comparing the similar triangles CDE and AFC, we get

$$\frac{DE}{FC} = \frac{CD}{AF}$$

or

$$DE = \frac{FC \times CD}{AF} = \frac{BD \times CD}{AB - CD},$$

which is the rule for the gnomonic shadow due to a lamp post.

Next, we find the tip of the gnomonic shadow from the lamp-post and height of the latter.

छायागुणितं छायाग्रविवरमूनेन भाजितं कोटी।
शङ्कुगुणा कोटी सा छायाभक्ता भुजा भवति ॥

This slóka means, "When there are two gnomons of equal height in the same direction from the lamp post, multiply the distance between the tips of the

shadows of the two gnomons by the larger (or shorter) shadow and divide by the larger shadow diminised by the shorter one. The result is the upright, i.e., the distance of the tip of the larger or shorter shadow from the foot of the lamp-post. The upright multiplied by the height of the gnomon and divided by the larger (or shorter) shadow gives the base, i.e, the height of the lamp post."

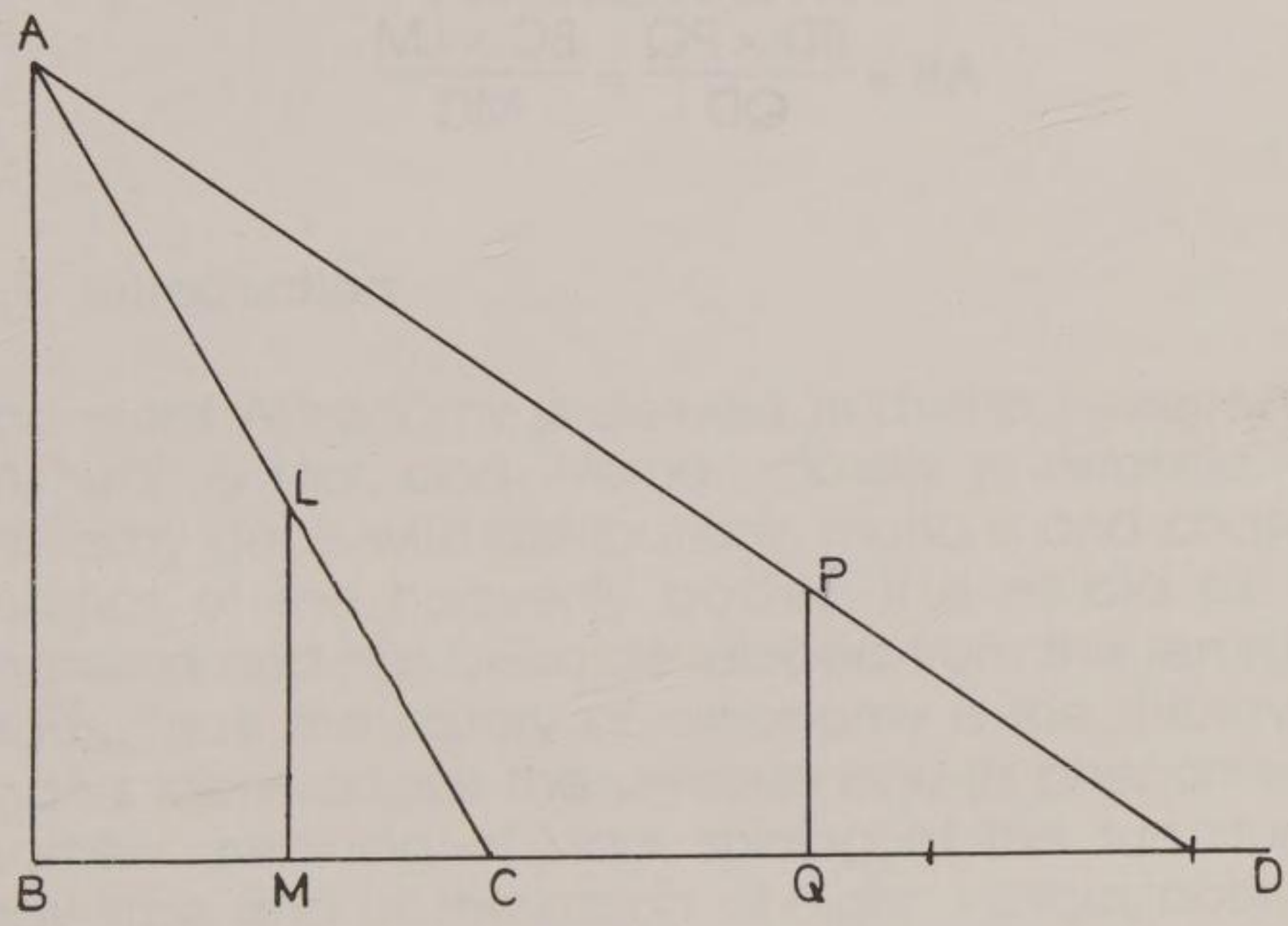


Fig. 6

In Fig. 6, AB is the lamp-post, BC or BD is the upright and LM and PQ are the gnomons of equal height.

We have

$$\frac{AB}{PQ} = \frac{BD}{QD} \quad \text{(i)}$$

$$\frac{AB}{LM} = \frac{BC}{MC} \quad \text{(ii)}$$

Since PQ = LM, therefore,

$$\frac{BD}{QD} = \frac{BC}{MC} = \frac{CD}{QD - MC} \quad \text{(iii)}$$

(i), (ii) and (iii) together, gives

$$BD = \frac{CD \times QD}{QD - MC} \tag{1}$$

$$BC = \frac{CD \times MC}{QD - MC} \tag{2}$$

and

$$AB = \frac{BD \times PQ}{QD} = \frac{BC \times LM}{MC} \tag{3}$$

CHAPTER 5

CONTRIBUTIONS OF ĀRYABHATA IN ASTRONOMY

5.1 Introduction

The word Astronomy is derived from the Greek word 'Astron,' a star, and 'Nemo,' classify or arrange. Astronomy deals with distributions, motions and characteristics of the heavenly bodies. It is as old as the mankind and has been developed from the remotest ages. Thus, the history of astronomy is the 'history of man's ideas above the universe and its phenomena, namely, twinkling of stars, shining of the sun during day-time and of the moon at night, various positions of planets, etc.' Ancient people must have observed these phenomenas with surprise. Gradually, they tried to know the reasons of these happenings and the occurrence of the sun and the moon and other unusual phenomenas stimulated them for closer study.

Astronomy is one of the oldest science cultivated by mankind and it developed as a science when the people learnt the methods of calculation and systematic observations of heavenly bodies began. It further developed as a perfect science through observations and experiments. G. Abetti observes, "Astronomy in various branches—theoretical, observational, experimental—continuously broadens its field of action and

more and more makes use of and co-operates with the related sciences." The theoretical prediction of eclipses of the sun and the moon and the motions of the planets are the greatest achievements in Astronomy.

A large number of works on mathematics and astronomy was written in India. during the period between 500 AD and 1200 AD. The earliest work on Hindu astronomy 'the *Āryabhatiya*' was written by Āryabhata in 499 AD. Āryabhata was followed by Varahamihira, Bhaskara I, Brahmagupta, Lalla, and others. All these astronomers created the science of Indian astronomy based on observations of heavenly bodies. They established the astronomical constants and constructed several instruments which were mostly used by them in observing the heavenly bodies and in formulating the astronomical principles. Astronomy made a remarkable progress during this period called as 'Golden period'.

Prof. Billard [5] says, "Above all, Indian history shows why Āryabhata was the leading figure of Indian astronomy and, even when introducing speculation into scientific knowledge, how he was a great astronomer and will soon be a great figure in the history of astronomy, in the history of sciences."

It is true that Āryabhata has played a magnificent and excellent role in the development and evolution of Indian astronomy. Prof. Shukla [25] has rightly called him the maker of scientific Indian astronomy. He says, "In the fifth century of the Christian era, when the primitive astronomy, taught in the five earlier *Siddhantas* which were studied in those days, failed to provide accuracy in astronomical calculations and there was a general complaint of divergence between astronomical prediction and observation, with the result that peoples faith in the words of astronomers was badly shaken, it was Āryabhata who came to the

rescue of the science. Āryabhata remodelled astronomy on a scientific basis, provided it with modified parameters, new tools and techniques and more accurate methods of calculations and became the saviour of astronomy."

An ancient saying goes,

"सिद्धान्तपंचकविधावपि दृग्विरूद्ध
मौढयोपरागमुखखेचर चार कतृत्तौ।
सूर्यः स्वयं कुसुमपुर्यभवत् कलौ तु,
भूगोलवित् कुलप आर्यभटाभिधान ॥"

It translates as, "When the methods of the five *Siddhantas* began to yield results conflicting with observed phenomena, such as the settings of the planets, the eclipses, etc., there appeared in the *Kali* age, at Kusumpura, *Surya* himself in the guise of Āryabhata, the *Kulapa*, who was well versed in astronomy."

Aryabhata was, no doubt, the founder of Indian astronomy and according to Thibaut, he was" the oldest of scientific Indian astronomers." His contributions in astronomy are enormous. We shall now briefly discuss his achievements and his scientific approach to astronomy.

5.2 Main Contributions

As we have said earlier, one of the greatest theories of Āryabhata is the one about the motion of the earth, its shape, size and other aspects. Because of this unorthodox theory, Āryabhata was the subject of heavy criticism for a long period of time. People could not, in fact, understand his theory about the earth. Even at that time, Āryabhata knew that the earth was round (not flat) and shine due to the Sun. In the fifth *slóka* of the *gola-pada*, he says,

"भूग्रहभानां गोलार्धानि स्वच्छायया विवर्णावि।
अर्धानिं यथासारं सूर्याभिमुखानि दीप्यन्ते ॥"

In the next *slóka*, he says,

"वृत्तभपञ्जरमध्ये कक्ष्या परिवेष्टितः खमध्यगतः।
मृज्जल शिखि वायुमयो भूगोलः सुर्वतो वृत्तः ॥"

It translates as, "The globe of the earth stands in space at the centre of the circular frame of the asterisms, surrounded by the orbits, it is made of water, earth, fire and air and is spherical. Further, he not only talks about the shape of the earth, but also of its construction. He narrates that the Earth is not stationary but moves around the Sun, which is clearly stated in the following ninth slóka of the *gola-pada*.

"अनुलोमगति नौर्रथ पश्यत्यचलं विलोमगं यद्धत्।
अचलानि भानि तद्वत समपश्चिमगानि लंकायाम् ॥"

It translates as, "Just as a man, sitting in a boat going forward, sees a stationary object moving backward, so at Lanka (a hypothetical place of 0 latitude and longitude) the immovable stars appear to move westward."

The following *slóka* is also believed to be due to Āryabhata.

भपंजरः स्थिरो भूखे आवृत्यावृत्य प्रतिदैवसिकौ।
उदयास्तगयौ सम्पादयति ग्रहनक्षत्राणाम् ॥

This translates as, "The structure of the asterisms is stationary. It is the Earth itself which, making a rotation everyday, causes the rising and setting of the stars and planets. Kamalakara (1658 AD) believed that the following *slóka* was also due to Āryabhata."

"स्थिरं प्रत्यकचलं भाति खस्थं प्राग्भूभ्रमान्नृणाम्।
प्रवहो व्यर्थ इत्यार्यमटोक्तेः श्रृण्हिोन्तरम् ॥"

The translation of this *slóka* is as follows. "On account of the eastward motion of the earth, the stationary objects in the sky appear to men as moving westwards. The idea of the provector wind (held by the ancients) is of no use."

As is already said, Āryabhata and his theory of rotation of the Earth were much criticised even upto 1000 AD.

"जानात्येकमपि यतो आर्यभटो गणितकाल गोलानाम्।
न मया प्रोक्राति ततः पृथक दूषणान्येशषाम् ॥"

In the above *slóka*, Brahmagupta went to the extent of saying that Aryabhata did not know even a bit of mathematics and astronomy and, therefore, he (Brahmagupta) did not want to count his shortcomings. Sripati rejected the theory of Āryabhata, he says

"यद्येवमन्बरचरा विहगाः स्वनीडमासादयन्ति न खलु भ्रमणेनके ङपि,
किंचाम्बुदा अपि न भूरि पयोमुचः स्युदेर्शष्य पूर्वगमनेन चिराय हन्त ॥"

The above *slóka* means that if the Earth is rotating, then the birds cannot come back to their nests and the clouds cannot give more water.

Somesvara also criticised Āryabhata's theory of the rotation of the Earth. He said, "This is a false view. For, if the earth had a motion, the world would have been inundated by the oceans, the tops of the trees and places would have disappeared, having been blown away by the storms caused by the velocity of the earth, and the birds, etc., flying in the sky would never have returned to their nests. So, there exists not a

single trace of the earth's motion. Hence, this stanza must be interpreted in another way."

Thus Somesvara provided the interpretation of the ninth *slóka* in the *gola-pada* of the *Āryabhatiya*, assuming the earth to be stationary. Āryabhata touched different aspects of astronomy. He provided such accurate approximations in astronomy that it is still a mystery, even today as to how he computed these values, though he has in certain places, indicated his method.

P.C. Sengupta [23] believes that "the scientific Hindu Astronomy was created by Āryabhata (476 AD)." Further, he says, "it was he who was the originator of the Epicyclic Astronomy in India." Āryabhata's contributions to Astronomy, some of which have been discussed in this chapter, evidently establish that he gave some new theories and contradicted certain old ones. At present Astronomy is considered as one of the branches of mathematics. Therefore, it is essential to deal with, at least, the outlines of astronomical principles in order to understand Āryabhata's contributions to Astronomy.

We can better appreciate Āryabhata's skill in calculating astronomical parameters if we look at the following table.

Table 10
Mean motion of the planets

Planets	Revolutions in 43,20,000 years	Sidereal peiod of one revolution in terms of days, according to:		
		Āryabhata	Ptolemy	Modern
Sun	43,20,000	365.25868	365.24666	365.25636
Moon	5,77,53,336	27.32167	27.32167	27.32166
Moon's apogee	4,88,219	3231.98718	3231.61655	3232.37543

Moon's asc. node	2,32,226	6794.74951	6796,45587	6793.39108
Mars	22,96,824	686.99974	686,94462	686.9797
Mercury	1,79,37,020	87.96938	87.96935	87.9693
Jupiter	3,64.224	4332.27217	4330.96064	4332.75637
Venus	20,22,388	224.69814	224.69890	224.7008
Saturn	1,46,564	10,766.06465	10749.94640	10759.201

Table 11
Diameters and distances of planets in *Yojunas*

	Āryabhata	*Āryabhata-Siddhanta*
Earth's diameter	1050	1600
Sun's diameter	4410	6480
Moon's diameter	315	480
Sun's distance	4,59,585	6,89,358
Moon's distance	34,377	51,556
Circumference of the sky/Revolutions of the moon	21,600	32,400

The following division of time is also due to Āryabhata:

1 day of *Brahma* = 14 *Manus* or 1008 *yugas*,
1 *manu* = 72 *yugas*
1 *yuga* = 43,20,000 years.

He rejected many other divisions already known before him.

Āryabhata worked on the motion and position of different planets. He said that a true planet moved in eccentric circles or in epicycles. To quote from the

Āryabhatiya. "All planets move on their (mean) orbits and their eccentric circles with their own (mean) velocities, eastwards (i.e anti-clockwise) from their apogees and westwards (i.e, clockwise) from their *sighroccas*." Āryabhata further said, "The radius of a planet's epicycle is the same as the distance between the centre of the planets eccentric circle and the earth. The planet which is situated on the circumference of its epicycle, moves with its mean velocity."

There are evidences which show that Āryabhata wrote at least one more book on astronomy, namely, the *Āryabhata-Siddhanta*.' This assertion is based upon the writings of Varahmihira (587 AD), Brahmagupta (628 AD), Bhaskara I (629 AD), Govindaswami (C.800–850 AD), Mallikarjuna Suri (1178 AD), Ramakrisna Aradhya (1472 AD), Maithila Candesvara, Bhudhara (1572 AD) and Tamma Yajva (1599 AD). Though we are not going into the details of the *Āryabhata-Siddhanta*, it is worth mentioning that this book was popular and studied throughout India, and that its popularity was at its peak in the 7th century AD. However, the *Āryabhatiya* and the *Āryabhata-Siddhanta* differ from each other in many respects from the astronomical point of view, as we can see in Table II given above for a detailed discussion, refer to Shukla [25]. Besides these two works, Āryabhata made other contributions also. In the words of A.K. Bag [3], "He wrote another work on the *artharatrika* system, which was responsible for influencing north-west India, Iran and the early Abhasid Caliphate. Besides these two works, some 34 verses on instruments, written by Āryabhata I, are recorded by Ramkrisna Aradhya (1472 AD), the commentator on the *Surya-Siddhanta*.

The following are some remarks of Shukla [25]: "Āryabhata holds a unique position in the galaxy of Indian mathematicians and astronomers whose works were studied through the ages like scriptures, and util-

ized in the construction of *pancangas* from Kashmir to Kerala. His contributions in algebra, trignometry and astronomy are of outstanding merit and of fundamental importance. It is he who taught for the first time the theory of the indeterminate equation, of the first degree in algebra, the value of π correct to four decimal places, the methods of computing the sine-table in trigonometry and the theory of the Earth's rotation in Astronomy. It is he who laid the foundation of the so-called *Siddhantic* astronomy and provided it with new tools and techniques, more accurate parameters and better methods of observation and computation."

In order to demonstrate the motion of the heavenly bodies, the Hindu astronomers make use of spheres constructed by means of a circle made of flexible wooden sticks or bamboo strips. These correspond to the celestial sphere of modern astronomy, and are called *gola*. One such *gola* is *Bhagola* which is supposed to be centred at the earth's centre. It is used to demonstrate the motion of the sun, the moon and the planets in their orbits. The principal circles of this sphere are (1) the celestial equation (2) the ecliptic (3) the orbits of the moon and the planets and (4) the day-circles, etc.. The *gola*, which is supposed to be centred at the observer is called *Khagola*, which is fixed in position, is used to demonstrate the diurnal motion of the heavenly bodies. The principal circles of this sphere are (1) the horizon (2) the meridian (3) the prime vertical (4) the six o'clock circle, etc.

5.3 Bhagola

मेषादेः कन्यान्तं सममुदगपमराडलार्धमपयातम्।
तौल्यादे र्मीनान्तं शेषार्ध दक्षिणेनैव ॥

This *slóka* translates as, "One half of the ecliptic,

running from the beginning of the sign of Aries to the end of the sign of Virgo, lies obliquely inclined (to be equator) northwards. The remaining half (of the ecliptic), running from the beginning of the sign of Libra to the end of the sign of Pisces, lies (equally inclined to the equator) southwards."

"ताराग्रहेन्दुपाता भ्रमन्त्यजस्त्रमपमण्डलेऽर्कश्च।
आर्काच्च मराइलार्धे भ्रमति हित स्मिन-क्षितिच्छाया ॥"

The nodes of the star planets (Mars, Mercury, Jupiter, Venus and Saturn) and of the Moon incessently move on the ecliptic, so also does the Sun. From the Sun, at a distance of half a circle, the shadow of the Earth moves thereon.

The nodes of a planet are the two points where the orbit of the planet intersects the ecliptic. The point where the planet crosses the ecliptic in its southerly course is called the "descending node."

अपमण्डलस्य चन्द्रः पाताद् यात्युत्तरेण दक्षिणातः।
कुजगुरु कोणाश्चैवं शीघ्रोच्चेनापि बुधशुक्रौ ॥

"The Moon moves to the north and to the south of the ecliptic from its (ascending and descending) nodes. So also do the planets Mars, Jupiter and Saturn. The motion of the *sighroccas* of Mercury and Venus is similar."

In reference to the above stanza Prthudaka (860 AD) says; "The Moon's (celestial) latitude at the place occupied by it is as much as that of Mercury or Venus at its *Sighrocca*." This is so because the revolution-number of the node (in the case of Mercury and Venus) is the sum of the revolution-numbers of the planet's node and the planet's *sighra* anomaly i.e., Sighrocca minus planet. This was written by Bhaskara II (1150 AD). However, the correct explanation of this

is that Mercury and Venus revolve round the sun with the velocity of their *sighroccas*, and so the (celestial) latitudes of Mercury and Venus are really the latitudes of their *sighroccas*. The following is the list of rules which are applied in the instructions of the text:

1. In the case of the Moon

$$R \sin (\text{latitude}) = \frac{R \sin (M - N) \times R \sin i}{H},$$

where M and N are the true longitudes of the moon and its ascending node, i is the inclination of the Moon's orbit to the ecliptic, and H is the Moon's true distance (called *mandakarna*).

2. In the case of Mars, Jupiter and Saturn

$$R \sin (\text{latitude}) = \frac{R \sin (P - N) \times R \sin i}{D},$$

where P and N are the longitudes of the planet and its ascending node, i is the inclination of the planet's orbit to the ecliptic, and D the distance of the planet from the Earth.

3. In the case of Mercury and Venus

$$R \sin \ (\text{latitude}) = \frac{R \sin (S - N) \times R \sin i}{D},$$

where S and N are longitudes of the planet's sighrocca and ascending node, i is the inclination of the planet's orbit to the ecliptic and D is the distance of the planet from the Earth.

According to Bhaskara I, these formulas are not correct and the correct formula for the celestial latitude of a planet is

$$R \sin \ (\text{latitude}) = \frac{R \sin (Q - N) \times R \sin i}{D},$$

where Q is the heliocentric longitude of the planet.

5.4 Khagola

पूर्वापरमधऊर्ध्व मण्डलमथ दक्षिणोत्तरं चैव।
क्षितिजं समपार्श्वस्थं भानां यत्रोदयास्तमयौ ॥

This *slóka* translates as, "The vertical circle which passes through the east and west points is the prime vertical, and the vertical circle passing through the north and south points is the meridian. The circle which goes by the side of the above circles (like a girdle) and on which the stars rise and set is the horizon."

In view of the fact that the centre of the *khagola* is at the observer lying on the surface of the Earth, the horizon is evidently the apparent or sensible horizon and not the rational horizon.

पूर्वापरदिग्लग्नं क्षितिजादक्षाग्रयोश्च लग्नं यत्।
उन्मराडलं भवेत्त् क्षयवृद्धि यत्र दिवस निशोः ॥

The *slóka* can be translated as, "The circle which passes through the east and west points, and meets the median above the north point and below the south point at distances equal to the latitude (of the place) from the horizon, is the equatorial horizon (or six o'clock circle) on which the decrease and increase of the day and night are measured."

Further, Aryabhata mentions that the centre of the *Khagola* lies at the observer or the position of the observer forms the centre of the *Khagola*. The *Gola-yantra* is the representation of the *Bhagola*.

The method used by Āryabhata, and described by the commentator Suryadeva, for rotating the sphere (Bhagola) at the rate of one rotation per twenty four hours is given below:

"Having set up two pillars on the ground, one

towards the south and the other towards the north, mount on the ends of the iron needle (rod). In the holes of the sphere, at the south and north poles, pour some oil, so that the sphere, may rotate smoothly. Then, underneath the west point of the sphere, dig a pit and put into it, a cylindrical jar with a hole at the bottom as deep as the circumference of the sphere. Fill it with water. Then, having fixed nail at the west point of the sphere and having fastened one end of a string to it, carry the string downwards along the equator towards the east point, stretch it upwards, carrying it to the west point (again), and then fasten to it a dry hollow gourd (appropriately) filled with mercury. Place it on the surface of water inside the cylindrical jar underneath. Then open the hole at the bottom of Jar so that with the outflow of water, water inside the Jar. Consequently, the gourd, due to the weight of mercury within it, does not leave the water and pulls the sphere westwards. The outflow of water should be manipulated in such a way that in 30 *ghatis* (= 12 hours) half the water of the jar flows out and the sphere makes one half of a rotation. Similarly, in the next 30 *ghatis* the entire remainder of water in the Jar flows out, the gourd reaches the bottom of the jar and the sphere performs one complete rotation. This is how one should, by using one's intellect, rotate the sphere keeping pace with time."

5.5 Spherical Astronomy

दृग्गोलार्धकपाले ज्यार्धेन विकल्पयेद् भगोलपार्धम्।
विषुवज्जीवाक्षभुजा तस्या स्त्ववलम्बकः कोटिः ॥

The above *slóka* translates as, "Divide half of the *Bhagola* lying in the visible half of the *Khagola* by

means of Rsines (so as to form latitude triangles). The Rsine of the latitude is the base of the latitude triangle. The Rsine of the colatitude is the upright of the same (triangle).

A right-angled plane triangle, whose sides are proportional to Rsin θ, Rcos θ and R, where R (= 3438) is the radius of the *Bhagola*, is called a latitude triangle. The latitude triangles play an important role in the solution of the spherical triangles in Indian astronomy. For, a number of results in astronomy are obtained simply by comparing two latitude triangles. Because of this importance of the latitude triangles, Āryabhata II (C. 950 AD) and Bhaskara II (1150 AD) have given a list of such triangles in their works.

Āryabhata gave a method for finding the radius of the day circle. It is as follows:

"Subtract the square of Rsine of the given declination from the square of the radius and take the square root of the difference. The result is the radius of the day circle, whether the heavenly body is towards the north or towards the south of the equator."

This implies that

$$\text{the day radius} = \sqrt{R^2 - (R \sin \delta)^2} \tag{1}$$

$$R \sin \delta = \frac{R \sin \lambda \times R \sin 24^\circ}{R} \tag{2}$$

λ and δ being, respectively, the Sun's tropical longitude and the declination of the heavenly body.

Aryabhata does not state formula (2) for finding R Sin δ . This can be obtained by the rule of three as follows:

"When the Rsine of the Sun's tropical longitude is equal to R, the R Sine of the Sun's declination is equal to Rsin 24°, then what will be the value of the Rsine of the Sun's declination when Rsine of the Sun's tropical longitude has the value Rsin λ? The result is Rsin δ."

Let α, β and γ denote the right ascensions of one sign, two signs and three signs, respectively, and δ_1, δ_2 and δ_3 the declinations at the last points of the signs Aries, Taurus and Gemini, respectively.

Then,

$$R \sin \alpha = \frac{R \sin 30^\circ \times R \cos 24^\circ}{R \cos \delta_1} \quad (1)$$

$$R \sin \beta = \frac{R \sin 60^\circ \times R \cos 24^\circ}{R \cos \delta_2} \quad (2)$$

$$R \sin \gamma = \frac{R \sin 90^\circ \times R \cos 24^\circ}{R \cos \delta_3} \quad (3)$$

On substituting $R \cos \delta_1 = 3366'$, $R \cos \delta_2 = 3218'$ and $R \cos \delta_3 = 3141'$ in (1), (2) and (3), respectively, we get $\alpha = 1670'$, $\beta = 3465'$ and $\gamma = 5400'$. Consequently,

the right ascension of Aries = α = 1670 respirations,

the right ascension of Taurus = $\beta - \alpha$ = 1795 respirations and

the right ascension of Gemini = $\gamma - \beta$ = 1935 respirations.

The right ascensions of Aries, Taurus and Gemini in the reverse order are the right ascensions of Cancer, Leo and Virgo; and the right ascensions of the first six signs, Aries etc. in the reverse order are the right ascensions of the last six signs, Libra etc. as shown in the following table:

Table 12
Right ascensions of the signs of the ecliptic

S.No.	Sign	Right ascension in respirations	Sign
1.	Aries	1670	12 Pisces
2.	Taurus	1795	11 Aquarius
3.	Gemini	1935	10 Capricorn
4.	Cancer	1935	9 Sagittarius
5.	Leo	1795	8 Scorpio
6.	Virgo	1670	7 Libra

Next, we discuss another method for deriving formulae (1) to (3) as given below:

Consider the celestial sphere for a place on the equator. Let the first point of Aries coincide with east point of the horizon; and let A be the last point of the sign Aries, AB the perpendicular from A on the east-west line, and AC the perpendicular from A on the plane of the horizon. Also, let G be the last point of the sign Gemini, GO the perpendicular from G on the east-west line and GM the perpendicular from G on the plane of the horizon.

Then on comparing the similar ΔS ABC and GOM, we have

$$AC = \frac{AB \times GM}{GO} = \frac{R \sin 30^\circ \times R \cos 24^\circ}{R}$$

Since $R \sin \alpha : R :: AC : R \cos \delta$, therefore,

$$R \sin \alpha = \frac{AC \times R}{R \cos \delta} = \frac{R \sin 30^\circ \times R \cos 24^\circ}{R \cos \delta}$$

Similarly, we can derive (2) and (3).

5.6 Eclipses of the Moon and the Sun

चन्द्रो जलमर्कोऽग्निः मृद्मूश्च्छायापि या तमस्तद्धि।
छादयति शशी सूर्य, शशिनं महती च भूच्छाया ॥

This *slóka* translates as, "The Moon is water, the Sun is fire, the Earth is earth and what is called Shadow is darkness (Caused by the Earth's Shadow). The Moon eclipses the Sun and the great Shadow of the Earth eclipses the Moon.

स्फुटशशिमासान्तेऽर्क पातासन्नो यदा प्रविशतीन्दुः।
भूच्छायां पक्षान्ते तदाधिकोनं ग्रहणमध्यम् ॥

The above *slóka* translates as, "When at the end of a lunar month, the Moon, lying near a node (of the Moon), enters the Sun or at the end of a lunar fortnight, enters the Earth's Shadow, it is more or less the middle of an eclipse (a solar eclipse in the former case and a lunar eclipse in the latter case)."

Āryabhata explains methods of finding the length of the Earth's Shadow and Earth's Shadow at the Moon's distance in the thirtyninth and fortieth *slókas* of the *Gola-pada*, respectively.

Multiply the distance of the Sun from the Earth by the diameter of the Earth and divide (the product) by the difference between the diameters of the Sun and the Earth. The result is the length of the Shadow of the Earth (i.e., the distance of the vertex of the Earth's Shadow) from the diameter of the Earth (i.e. from the centre of the Earth). This means:

The length of the Earth's Shadow

$$= \frac{\text{Sun's distance} \times \text{Earth's diameter}}{\text{Sun's diameter} - \text{Earth's diameter}}$$

The Hindu method, which is called "the lamp and shadow method," is used to derive the above mentioned formula. It is as follows:

In the figure given below, S is the centre of the Sun and E that of the Earth. SA and EC are drawn perpendicular to SE and denote the semi-diameters of the Sun and the Earth, respectively. BC is parallel to SE. V is the point where SE and AC produced meet each other.

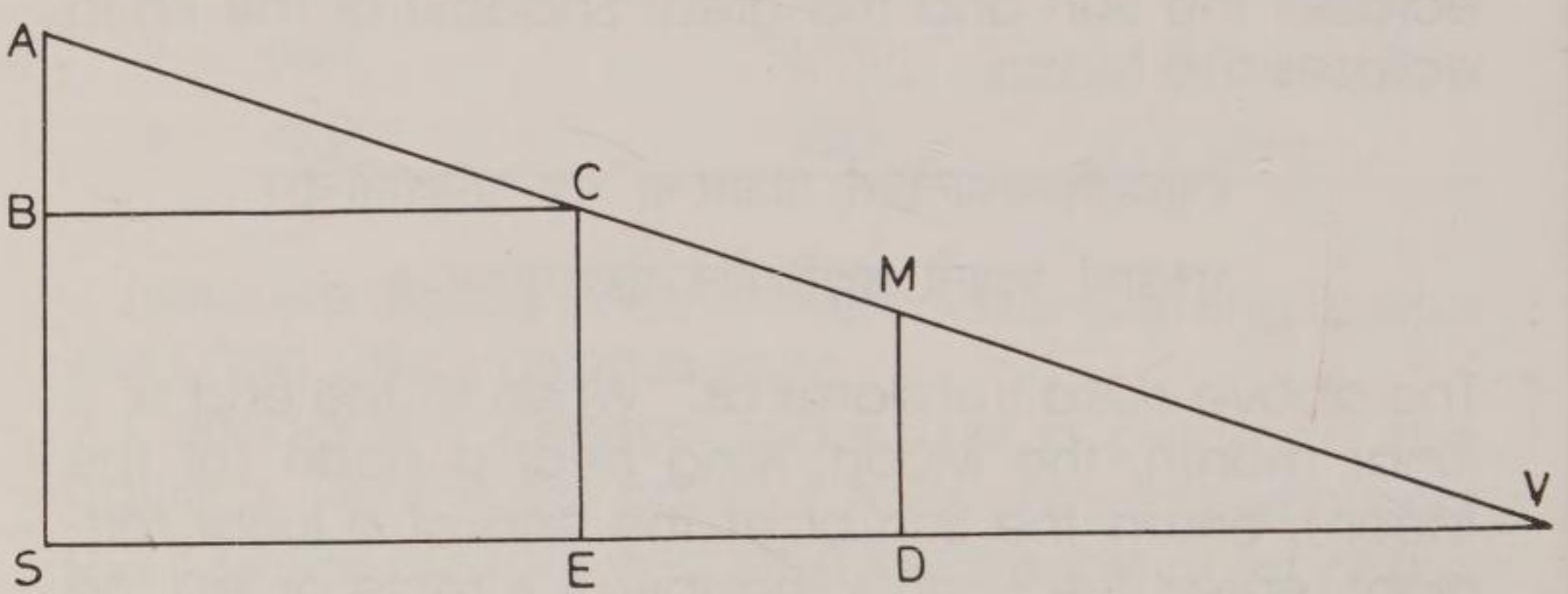


Fig. 6

Hindu astronomers compare SA with a lamp post, EC with a gnomon and EV with the shadow cast by the gnomon due to the light of the lamp. Consequently, they call EV 'the length of the Earth's Shadow from the diameter of the Earth.'

The triangles CEV and ABC are similar. Therefore,

$$\frac{EV}{EC} = \frac{BC}{AB} = \frac{SE}{SA - EC}$$

$$\therefore \quad EV = \frac{SE \times EC}{SA - EC} = \frac{SE \times 2EC}{2SA - 2EC}$$

i.e., Length of the Earth's Shadow

$$= \frac{\text{Sun's distance} \times \text{Earth's diameter}}{\text{Sun's diameter} - \text{Earth's diameter}}$$

Multiply the difference between the length of the Earth's Shadow and the distance of the Moon by the Earth's diameter, and divide (the product) by the length of the Earth's Shadow, the result is the diameter of the *Tamas* (i.e, the diameter of the Earth's Shadow at the Moon's distance) It means:

Diameter of *Tamas*

$$= \frac{\text{(Length of Earth's Shadow} - \text{Moon's distance)} \times \text{Earth's diameter}}{\text{Length of Earth's Shadow}}$$

In the above Fig. 6, M is the position of the Moon when it is just on the point of entering the Earth's Shadow. MD is perpendicular to SE, so it denotes the semi-diameter of the Earth's Shadow at the Moon's distance, i.e., the semi-diameter of the *Tamas*.

The Triangles MDV and CEV are similar.

$$\therefore \quad \frac{MD}{DV} = \frac{CE}{EV}$$

$$\text{or} \quad MD = \frac{(EV - ED) \times CE}{EV}$$

$$\text{or} \quad 2MD = \frac{(EV - ED) \times 2CE}{EV} = \frac{(EV - EM) \times 2CE}{EV}$$

approximately.

i.e., Diameter of Tamas

$$= \frac{\text{(Length of Earth's Shadow} - \text{Moon's distance)} \times \text{(Earth's diameter)}}{\text{Length of Earth's Shadow}}$$

The colour of the moon during eclipse is described

by Āryabhata in the forty sixth slóka of the *Āryabhatiya*, as follows:

"At the beginning and end of its eclipse, the Moon (i.e, the obscured part of the moon) is smoky; when half obscured, it is black, when totally obscured, it tawny and when far inside the shadow, it copper-coloured, with blackish tinge."

In case of a solar eclipse, the obscured part of the sun looks black at every phase of the eclipse.

5.7 Planets Determined from Observations

क्षितिरवियोगाद दिनकृद् रवीन्दुयोगात् प्रसाधितश्चेन्दुः।
शशितारााग्रहयोगात् तथैव तारााग्रहाः सर्वे ॥

The translation of this slóka is, "The Sun has been determined from the conjunction of the Earth and the Sun, the Moon from the conjunction of the Sun and the Moon and all the other planets from the conjunctions of the planets and the Moon."

According to the commentator Somesvara, the method of finding the revolutions of the Sun, Moon and the Planets in a *yuga* by observing the conjunctions is as follows:

(1) Revolutions of the Sun

Let the number of conjunctions of the Sun and the Earth (the number of civil days) in one solar year be *x*. Then the number of revolutions of the sun in a *Yuga* = *X/x*, where *X* denotes the number of civil days in a yuga.

(2) Revolutions of the Moon

Let the number of conjunctions of the Sun and the

Moon in one solar year be *x*. Then, the number of revolutions of the Moon in a *yuga* = $(x+1)S$, where *S* denotes the number of solar years in a *yuga*.

Alternatively, let the number of risings of the Moon in one solar year by *y*. Then the number of revolutions of the Moon in a *yuga* = $E - yS$, where *E* denotes the number of rotations of the earth and *S* the number of solar years in a *yuga*.

(3) Revolutions of the planets (Mars, Mercury, Jupiter, Venus and Saturn) round the Earth

Let the number of conjunctions of a planet and the Moon in one solar year be *z*. Then, the number of revolutions of the planet in a *Yuga* = $M - zS$, where *M* denotes the number of revolutions of the Moon and *S* the number of solar years in a *yuga*.

(4) Revolutions of Sighroccas of Mercury and Venus

Let *C* denote the number of civil days in a *yuga*, and *d* the number of days between two consecutive inferior or superior conjunctions of the planet and the Sun. Then,

> *C/d* = the number of revolutions of the planet's *sighrocca* – the number of revolutions of the Sun in a *yuga*

It means that

> the number of revolutions of the planet's *sighrocca* in a *yuga* = *C/d* + Sun's revolution in a *yuga*.

R.C. Gupta described [15] Madhava's rule for finding the angle between the ecliptic and the horizon and *Āryabhatiya's* knowledge of it.

CHAPTER 6

RELEVANCE OF ĀRYABHATA TO MODERN MATHEMATICS AND SCIENCE

6.1 The Renaissance of Scientific Methodology Inspired by Āryabhata

According to some historian of science, Āryabhata was the first great scientist, astronomer and mathematician of ancient India. Before him there were many great *rishis* who made mathematical contributions to the *Vedas* and to the *Sulva Sutras*. The seven *Sulva Sutras* were named after seven great *rishis*, but these were essentially religious works, in which mathematics came in because of the need for construction of altars, while mathematics and astronomy together came in because these were needed for determining auspicious times for religious rituals.

Āryabhata also approached the subject in a religious spirit, but in a different manner. He was a great devotee of Lord *Brahma*, the creator of the universe. In fact, some persons believe that the knowledge contained in the *Āryabhatiya* was revealed to him by Lord *Brahma*, who was pleased with his great devotion. This may or may not be true, but it is true that it was his devotion to Lord *Brahma* that inspired him to

find but the truth about the universe created by *Brahma*. In fact, he wrote at one place, "Knowing the *Dasqutika Sutras,* the motion of the Earth and the planets on the celestial sphere, one attains the supreme *Brahma* after passing through the orbits of the planets and the stars."

In *Āryabhata's* view the goal of all learning was the attainment of the Supreme One and this could easily be achieved through astronomy. Since mathematics was needed in order to understand astronomy, it was also very important. He believed in the steps "Mathematics→Astronomy→Supreme Brahma," and he looked at mathematics and astronomy as the means to a much higher goal. This was different from the earlier attitude, namely, that mathematics was needed for the construction of altars and that mathematics and astronomy were subservient to astronomy for finding auspicious *mahurats* for rituals for propitiating gods and goddesses. His attitude was that it was necessary to investigate the truth about Nature in order to know the truth about its Creator.

The earlier attitude was to propitiate the force of Nature through rituals. Āryabhata's attitude was to understand Nature.

He also believed in his observations and in going wherever his observations led him. He established new values of astronomical parameters by careful observations. He did not accept any statement of any authority, however great that authority might be. This again, is the attitude of a true scientist.

He studied the available literature at that time and found many mathematical and astronomical results scattered in various religious books. He studied these, and corrected the errors that he found in them. He found some gaps and filled them, and also looked in new directions. He felt the need for consolidation of existing knowledge which he did systematically. In

fact, he thought that it was desirable to write a text-book giving both the earlier and the new results, and he prepared such a text-book.

He did not prove his results in the tradition of Greek mathematics, but he gave these in compact form in the Indian tradition. In just thirty three slókas dealing with mathematics, he considered geometrical figures and their properties, mensuration, interest, series, simultaneous and quadratic equations, linear indeterminate equations, square roots, cube roots, method of constructing sine tables, etc.

Similarly in just fifty slókas, he dealt with the motion of the Sun, the Moon and the planets on the celestial sphere. He described the various circles on this sphere and indicated the method of automatically rotating the sphere once in 24 hours to explain the motion of the Earth, Sun, Moon and planets. He discussed the motion of the celestial sphere, as seen by those on the equator and by those on the north and south poles, and gave rules regarding the various problems of spherical astronomy. He also dealt with eclipses and visibility of the planets.

Since Āryabhata was most probably the *Kulpati* of Nalanda University, his work had a big impact. In fact, Āryabhata ushered in a renaissance in Indian mathematics, astronomy and science, which resulted in a remarkable flourishing of science and technology in India in the succeeding centuries. It was not only his contributions to mathematics and astronomy that mattered, but also his scientific attitude, his attitude to learning and teaching and of not taking anything for granted, and his style of presentation that set the pace for future generations.

He had a number of disciples who continued his work, but his example was also followed by scientists in other fields, who were neither his disciples, nor the disciples of his disciples. Great mathematicians like

Brahmagupta, Mahavir, Acharya Bhaskaracharya and others were inspired by Āryabhata's remarkable contributions to mathematics and astronomy and they incorporated these contributions in their excellent text-books.

At the same time, mathematics prospered in India because we had a succession of excellent commentators who wrote excellent commentaries on the works of Āryabhata and other great scientists. In these commentaries they provided possible modifications and applications of mathematics, brought out the subtle points, and, what is even more important, provided *utpaties* or proofs of results.

Āryabhata, for the first time, secularised mathematics and astronomy in India and established these as intellectual disciplines in their own right. Previously they were handmaidens of religion, or rather, of religious rituals. This delinking of mathematics and astronomy from religion had taken place in Greece much earlier and this had led to rapid development of mathematics there.

It was the genius of Āryabhata that led him to discuss mathematics and astronomy as separate disciplines and this freedom gave flexibility, which allowed mathematics to develop rapidly in the succeeding centuries.

This did not mean that mathematics became abstract and lost touch with applications. In Greece, it was associated with the mechanics of Archimedes and the astronomy of Ptolemy, while in India it was associated with Āryabhata astronomy.

6.2 Some Lessons from Āryabhata's Life and Work

What is the relevance of Āryabhata to modern Indian mathematics and science? In other words, what

lessons can we get from his life and work, which may be used for the future development of mathematics and science in India?

1. We should be clear about our goals for research in mathematics and science. Should we do research in science and technology to get prestige and to rise in the estimation of western countries or should we do this research primarily in our own national interest? Āryabhata had an open mind, but he consolidated the existing knowledge in India and gave a new direction to scientific research here.

2. Āryabhata showed originality and creativity of a high order and this is what we need in a big way today.

3. Āryabhata was a great expositor. While it is important to do original and creative research, it is also important to present known results in a lucid, unified manner, so that more persons can be trained for high level research. Moreover, such expositions have to include the results of the latest researches in order to be exciting and stimulating to students and this was what Āryabhata did.

4. The object of research in mathematics and science is to understand Nature, so that we can use the forces of Nature for the benefit of mankind. These forces can best be understood not by *Poojas* but by hardwork and a dedicated pursuit of mathematics and science.

5. We should be completely accurate, if possible, but if we cannot get completely accurate results, we should be as accurate as possible. Thus, in Āryabhata's approach, the value of π was given as 3.1416, which was the best approximate value known anywhere in the world till that time. But what was more important was that he knew that it was an approximate value.

6. Āryabhata gave interesting and imaginative

problems to catch the imagination of his readers. We should also propose problems which may show both the beauty and utility of science and mathematics.

7. Āryabhata had sound knowledge of arithmetic, algebra and trignometry. He was not so strong in geometry, because he did not require this strength in his study of astronomy and because he was not studying mathematics for its own sake. The geometry that was known was enough for practical application. However, we must always go beyond the immediate applications in mathematics and science, otherwise we suffer. Indian mathematicians went beyond their immediate needs in algebra, but in geometry, they stopped as soon as their immediate needs were satisfied. On the other hand, the Greeks went beyond their immediate needs in geometry, but stopped in algebra when their immediate needs were satisfied. This shows that in mathematics and scientific research, we should be ready to go beyond immediate needs. However, we should not go too far in one direction at the expense of moving only short distances in other directions. Many mathematicians do not develop mathematics for immediate needs, but rather for needs which may or may not arise in future. We must give a high priority to develop in mathematics for solving our immediate problems of science and technology. At the same time, we must study mathematics for its own sake, but we must keep an appropriate balance between the two types of mathematics.

8. Āryabhata believed in experimental mathematics, intuitive mathematics, empirical mathematics and heuristic methods of mathematics. He, like other Indian mathematicians, did not use the axiomatic method. The Indian mathematicians, did not make much progress in geometry because they did not use the axiomatic method. On the other hand, the Greeks could not make much progress in algebra, because

they could not develop, at that time, a powerful axiomatic system for algebra and besides, were reluctant to study any system which could not be studied axiomatically. The Indians went far in algebra because they were interested in results and not, primarily, in the methods by which these results were obtained. We should use the axiomatic method as far as possible, but should not be so obsessed with it that we do not make any progress. For some areas in pure mathematics, the axiomatic method is essential, but for some areas of applications, heuristic methods are more useful. We should use a judicious mixture of methods. We should, in general, first use experimental methods to make quick progress and get results, and then use axiomatic methods to consolidate these results. Axiomatics without intuition may lead to barren results, while intuition without rigour may lead to results of uncertain validity.

9. We know that Āryabhata ushered in a renaissance in Indian mathematician. The modern mathematician must investigate the culturally, social, economic and political influences which favoured his playing this crucial role. It is said that the Gupta king Buddha Gupta came to the throne in the year of Āryabhata's birth and that Āryabhata grew up in his reign. Did Buddha Gupta create a climate in which scientific talent could find fulfilment, or was it the invigorating climate of Nalanda University which nourished Āryabhata's talent?

10. What was Āryabhata's knowledge of mathematics in other countries and how far did this knowledge influence him? Āryabhata made the bold statement that the stars are stationary and the earth rotates about its axis once in twenty four hours. He could make this bold statement without fear of persecution in India in those days, while Galileo suffered heavily for making a similar statement much later in

Europe. So, the atmosphere of free enquiry in India at that time was of much help to Āryabhata. The modern mathematician must investigate such problems in cooperation with historians of that period.

11. Even otherwise, modern mathematicians should give a higher priority to research on history of mathematics. They should investigate what conditions favoured the development of mathematics in the past and what conditions retarded its growth in India, what conditions favoured the communication of mathematical ideas to other countries and what conditions hindered and led to lack of communication? Indian mathematicians should study the history of Indian mathematics in a wide context as well as the history of mathematics in other civilizations.

12. We should study Āryabhata's mathematics and astronomy from the modern point of view in a comprehensive way. It does not matter that such a study reveals some weaknesses in his mathematics, since these are bound to be there when examined after 1500 years of great developments in mathematics. Such a study will, however, also bring out his strengths in a significant manner.

13. There is a lesson for us today that we must neither neglect the abstractions in mathematics, nor fail to consider the significant applications of mathematics. We must combine these two in a harmonious manner. Only then can a rapid development of mathematics, science and technology take place.

14. Another point that we notice in ancient Indian mathematics, and to some extent in Āryabhata's mathematics and astronomy, is that excellent gems of mathematics are sometimes placed side by side with very poor quality mathematics. We must learn to distinguish between good and bad mathematics and aim at the highest quality therein.

15. Āryabhata had the courage to question

established results. However, soon after his death, some of his disciples and others began to refer to him as *Bhagwan* Āryabhata and considered everything that he had said as sacred and beyond doubt. Later mathematicians wrote a large number of commentaries on the *Āryabhatiya* and examined the results critically. While we should have high regard for all our great mathematicians and scientists, we should be prepared to question their results, if we are not convinced. We should even be prepared to throw out some of their invalid and trivial results, retaining only their best results, this will prepare the structure of mathematics and science on the best results of the past.

16. In India, we could make rapid progress in arithmetic and algebra because we had a good system of notation, while the Greeks suffered significantly because they did not have a similar advantage. Thus, for making progress in science and mathematics, a suitable notation and suitable conceptual framework are very important. We should consistently re-examine our notations and framework and should not hesitate to replace these by better notations and concepts, if we can find them.

6.3 Concluding Remarks

Aryabhata was a great pioneer in mathematics and astronomy and we can all be proud of his great achievements. However, the greatest tribute we could pay him would be to value his spirit of free enquiry in search of truth at all costs, and try to emulate his great dedication to mathematics and science.

The detailed and diogistic study of various contributions of Āryabhata to the development of Astronomy and Mathematics confirms that he was the most

original innovator and great researcher of his time. His various research findings not only inspire the contemporaries, but also served as the daily intellectual food for the knowledge seekers in succeeding generation. His works influenced the Arabs who spread them to the other part of the world. In general Āryabhata is rated as one of the greatest mathematicians of all time and of all history.

Āryabhata's contribution to astronomy is original and very special. He was the first who constructed epicyclic astronomy. He discovered the causes of eclipses and scientific method of calculating the time of their occurrences. The fact that the earth rotates about its axis and also revolves round in Sun was given by Āryabhata. S.N. Sen says "unlike his great contemporary Varahamihira of encyclopaedic scholarship, Āryabhata was no more compiler, but an original thinker and innovator who boldly sought to reform astronomy on scientific lines." Thus there was none in ancient India who could complete him in knowledge and researches in astronomy. S.S Pillai observes, "If Kalidas is the Shakespeare of India, if Valmiki, the Indian Homer, Āryabhata may be called the Indian Newton." It is significant that he developed Indian astronomy without the aid of modern telescope or spectroscope and other astronomical instruments. He did the same for Indian astronomy as Ptolemy had done for the Greeks. So it can be said that the eminence of Āryabhata in India is similar to that of Ptolemy in Greece.

REFERENCES

1. Ansari, S.M.R.: *A comparative study of the mathematics of Āryabhata I and Al-Khwarizmi* — Celebrations of the 1500th Birth Anniversary of Āryabhata I, Indian National Science Academy, New Delhi, Page I, 1976.
2. Ayyangar, A.A.K.: *The Mathematics of Āryabhata* Quart. Journal Myth. Soc. 16, 1925-26, 158-79.
3. Bag, A.K.: *Āryabhata as he was known to later scholars* — Celebrations of the 1500th Birth Anniversary of Āryabhata I, Indian National Science Academy, New Delhi, page 21, 1976.
4. Bhanu Murthy, I.S: *A modern introduction to Ancient Indian Mathematics,* Wiley Eastern Limited, New Delhi, 1992.
5. Billard, R.: *Āryabhata and Indian Astronomy,* Indian Journals of History of Science, Vol. 12(2), 207-224, 1977.
6. Behari, R.: *Āryabhata as a Mathematician,* Indian Journals of History of Science, Vol. 12(2), 147-149, 1977.
7. Bhaskara I: *Maha-Bhaskariya and Laghu-Bhaskariya,* Anandashram Sanskrit Granthawali, 1945.

8. Boyer, C.B.: *A history of mathematics*, John Willey and Sons, New York, 1968.
9. Clark, W.E.: *The Āryabhatiya of Āryabhata*, the University of Chicago Press, Illinois, 1930.
10. Datta, B.: *Two Āryabhatas of Al-Biruni*, B.C.M.S., 17, 59-74, 1926.
11. Datta, B.B. and Singh, A.N.: *History of Hindu Mathematics*—A Source Book Reprinted, Asia Publishing House, Bombay.
12. Elfering, K.: The area of a triangle and the volume of a pyramid as well as the area of a circle and the surface of hemisphere in mathematics of Āryabhata I, Indian Journals of History of Science Vol. 12(2), 232-236, 1977.
13. Ganguly, S.K.: The Āryabhata's value of π Ann. Math. 27, 16-29, 1920.
14. Ganitanand: A few remarks concerning certain values of π in ancient India, Ganita Bharati 12, No. 1-2, 33-39, 1990.
15. Gupta, R.C.: On some mathematical rules from the *Āryabhatiya*, Indian Journals of History of Science, Vol. 12(2), 200-206, 1992.
16. Jha, Parmeshwar: *Āryabhata I and his contributions to mathematics*. The Bihar Research Society, Patna (1988).
17. Jha, S.K.; Jha, M.: A study of the value of π known to ancient Hindu and Jaina mathematicians, J. Bihar Math. Soc., 13, 38-44, 1990.
18. Kaye, G.R.: *Two Aryabhatas*, B.M., 10, 289-92, 1910.
19. Mohan, B.: *History of mathematics* (In Hindi), Hindi Samiti, U.P. Govt., Lucknow, 1965.
20. Mule Gunakar: *Great mathematicians of the world*, Raj Kamal Publications, New Delhi, 1992.
21. Prakash, S.: *Founders of Sciences in ancient India*, Research Institute of Ancient Scientific Studies, New Delhi.

22. Premi, S.: A project Report on history of ancient Indian mathematics, Department of Mathematics, Meerut University, Meerut.
23. Sengupta, P.C.: *The Āryabhatiya* (English version), J. Dept. Letters, Calcutta University, 16, 1-56, 1927.
24. Sharma, M.L.: *Āryabhata's contribution to Indian astronomy of Indian Journal of History of Science*, Vol. 12, 90-99, 1977.
25. Shukla, K.S.: *Āryabhata*, Indian National Science Academy, Bahadur Shah Jafar Marg, New Delhi.
26. Shukla, K.S. and Sarina K.V.: *Āryabhatiya of Āryabhata*, Indian National Science Academy, New Delhi, 1976.
27. Singh, A.N.: On the Indian method of root extraction, B.C.M.S., 18, 123-40, 1927.
28. Smith, D.E.: *History of mathematics*, Dover Publications, New Delhi, 1959.
29. Swarup, A.K. Bag and K.S. Shukla: Proceeding of the International Astronomical Union Collogium held in New Delhi, Cambridge University Press, 1987.
30. Upadhya, B.L.: *Ancient Indian Mathematics* (in Hindi), Vigyan Bharati, New Delhi, 1971.
31. Voladavsky, A.: *Mathematical achievements of Āryabhata, Indian Journal of History of Science*, Vol. 12(2), 167-72, 1977.
32. Yano Michio: *Three types of Hindu sine tables; Indian Journal of History of Science*, Vol. 12(2), 83-89, 1977.

(2)

Of Related Interest

PHYSICS IN ANCIENT INDIA

N.G. DONGRE

Historians have established that the Indian Civilization was one of the most advanced and sophisticated of all the early civilizations, so much so that its dynamic and progressive impact was felt in the evolution of other civilizations. This was so in many areas of learning. This work primarily deals with its impact on the field of Physics and, thereby, presents evidence which shows that scientific concepts, such as Newton's Laws of Motion were existent at that time. Not only were fundamental concepts existent, but they had been developed in a systematic and highly organized manner, following the observation, inference and conclusion procedure. The outcome of comprehensive study from available literature in *Sanskrit*, the work brings to fore the systematic procedural methodology followed by the ancient Hindus.

It draws mainly from *Vaisesika Darshan*, one of the six well-known systems of *Vedic* philosophy propounded by *Aulukya*, more popularly known as *Kanad*.

The work, collating from various sources, gives details pertaining to Physics and, in the process tries to draw a parallel between the scientific temperament which prevailed in earlier days and the one which evolved in relatively recent times.

It breaks entirely new ground and should be of interest to the scholars in the field.

This brave unique work sets us thinking and we are left wondering as to whether we are moving towards some sort of degeneration or rediscovery of what already existed centuries ago. The book, no doubt, makes interesting reading and, paradoxically enough, leaves us puzzled, yet excited.

PUBLISHING FOR ONE WORLD

NEW AGE INTERNATIONAL (P) LIMITED, PUBLISHI

Formerly WILEY EASTERN LIMITED

New Delhi • Bangalore • Bombay • Calcutta • Guwahati • Hyderabac

Lucknow • Madras • Pune • London • Bangkok

9 788122 407

W7-DCO-938